Lectures #4-5

Clebsch-Gordon coefficients vs 3j symbols

Irreducible tensor operators

Graphical representation

Wigner-Eckart theorem (again)

6j symbols

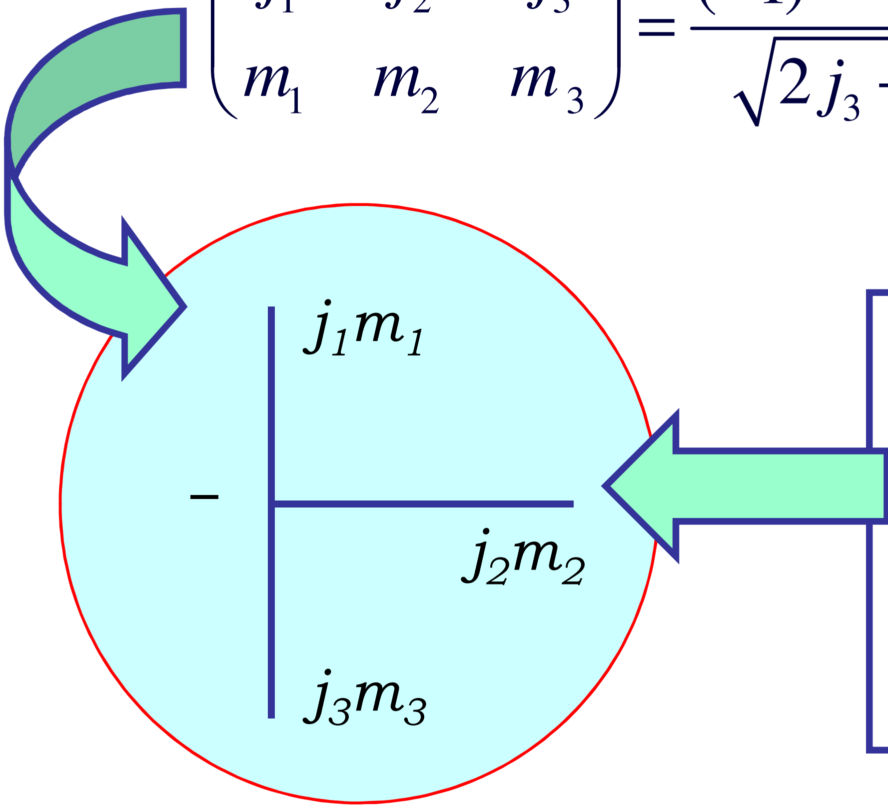
Chapter 1, pages 11-20, 99-102 of the Lectures on Atomic Physics

Atomic many-body theory, Lindgren & Morrison

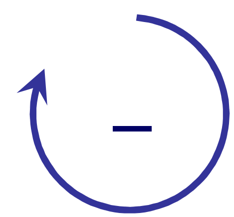
Three-j symbols vs. Clebsch-Gordan coefficients

$$m_1 + m_2 + m_3 = 0$$

$$\overbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}^{3-j\ symbol} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3+1}} \overbrace{\langle j_1 m_1 j_2 m_2 | j_3 - m_3 \rangle}^{Clebsch-Gordan\ coefficient}$$



Graphical representation
of the 3-j symbol
– means "read in clockwise direction"
(123 here)



Why change Clebsch-Gordan coefficients to 3-j symbols?

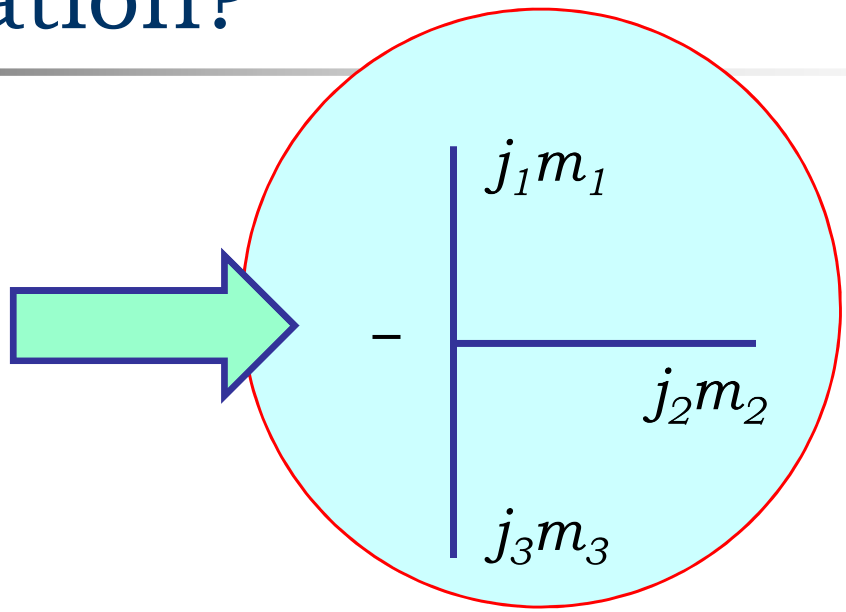
$$\overbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}^{3-j\ symbol} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \overbrace{\langle j_1 m_1 j_2 m_2 | j_3 - m_3 \rangle}^{Clebsch-Gordan\ coefficient}$$

3-j symbols have much easier symmetry relations with respect to interchanging 1,2,3 indices and m signs! Therefore, they are much easier to use in real calculations when one needs to sum over magnetic quantum numbers m!

Why do we introduce graphical representation?

3–j symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

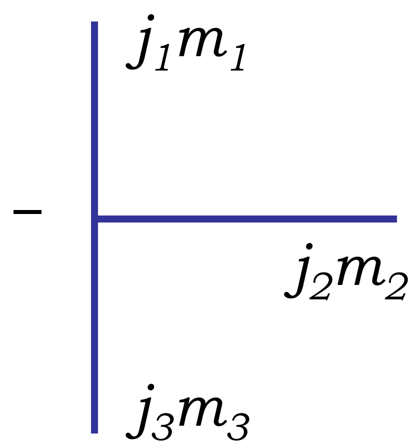


- In actual calculations, one may need to sum large number of 3-j symbols over large number of magnetic moments.
- It is easier to do it using diagrams than by writing out actual expressions.
- One is less likely to make a mistake using diagrams.
- It is also easier to trace and correct mistakes both owing to more compact form and use of “standard expressions”.

Three-j symbols: symmetry relations

$$\begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

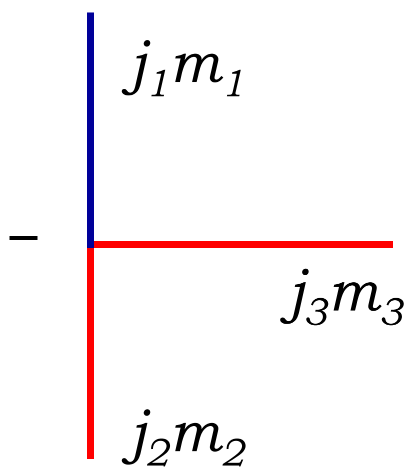
$312 - 231 - 123:$ *no change with even permutations*



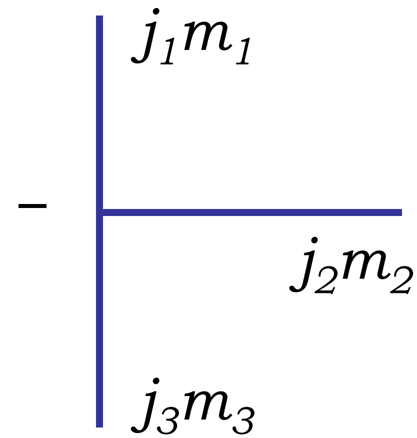
Three-j symbols: symmetry relations

132 – 213 – 321: *phase change with odd permutations*

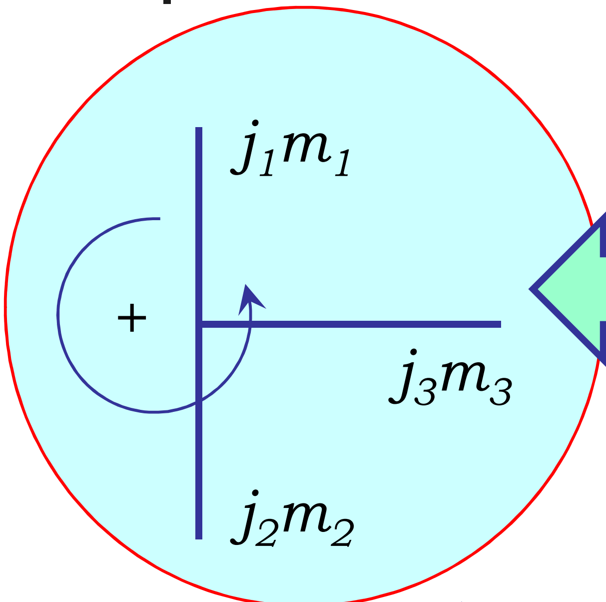
$$\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$



$= (-1)^{j_1+j_2+j_3}$



Graphical representations of 3-j symbols

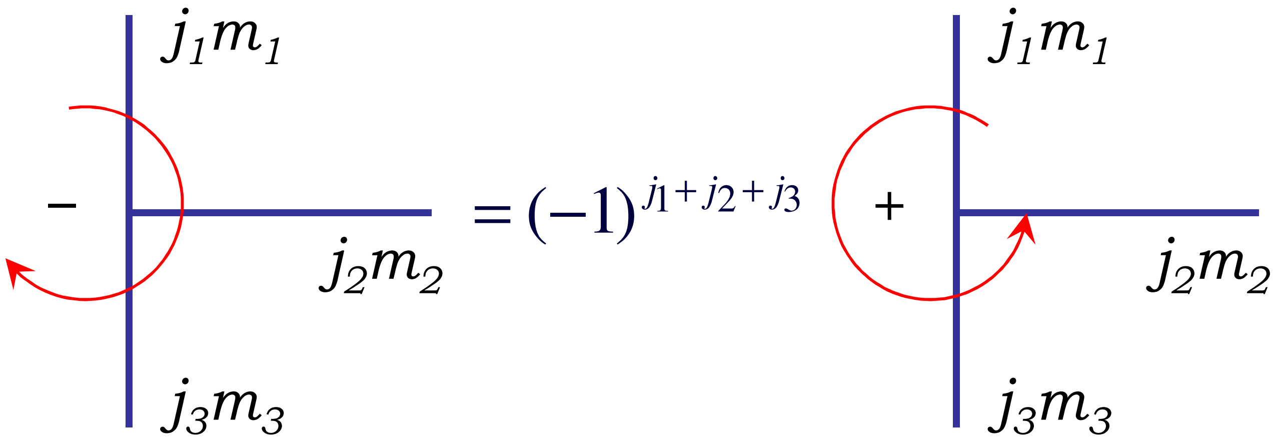


Graphical representation
of the 3-j symbol
+ means "read in counter-clockwise direction"
(123 here)

+

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Graphical representation of 3-j symbols: "+" vs. "–"



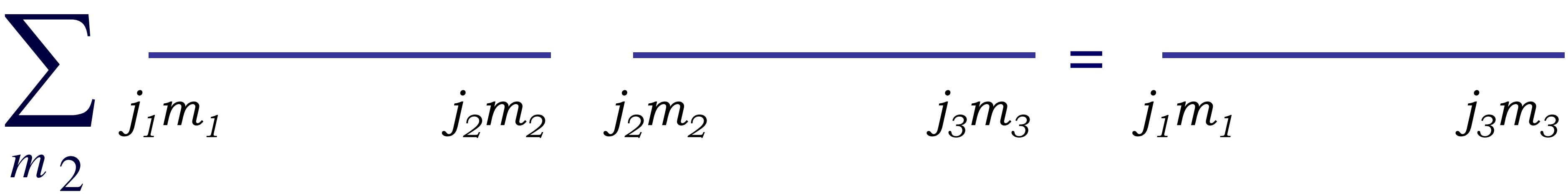
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

Lines and sums

$= \delta_{j_1 j_2} \delta_{m_1 m_2}$ ← This means that j and m stay constant within the same line

So how does one sum over magnetic quantum numbers m?

By simply joining the lines with the same m!



A note on integer quantities

2j, 2m, j-m and j+m are always integers

If j_1, j_2, and, j_3 satisfy the triangular condition

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2$$

$j_1 + j_2 + j_3$ is an integer

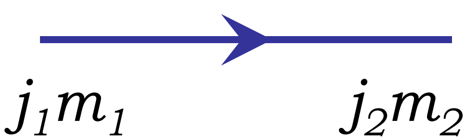
Why is this important?

$$(-1)^{4j} = (-1)^{4m} = (-1)^{2(j-m)} = (-1)^{2(j+m)} = (-1)^{2(j_1+j_2+j_3)} = 1$$

Arrows and minus signs

What if you have m with a minus sign?

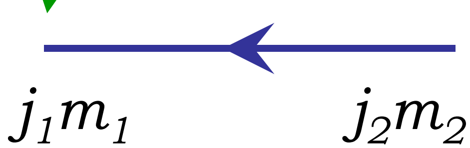
Then put an arrow on that line in the direction of m with a minus sign!



$= (-1)^{j_2 - m_2}$

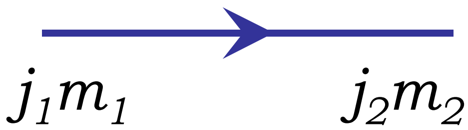
$= (-1)^{j_2 - m_2} \delta_{j_1 j_2} \delta_{m_1 - m_2}$

Note: arrow is in the direction of m with a minus sign

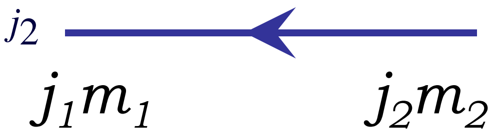


$= (-1)^{j_1 - m_1}$

How to switch the direction of an arrow?



$= (-1)^{2j_2}$



$$(-1)^{j_2 - m_2} \delta_{j_1 j_2} \delta_{m_1 - m_2}$$

$$(-1)^{j_1 - m_1} \delta_{j_1 j_2} \delta_{m_1 - m_2}$$

$$\downarrow$$

$$(-1)^{j_2 + m_2} \delta_{j_1 j_2} \delta_{m_1 - m_2}$$

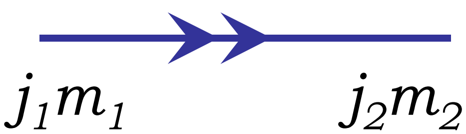
$$\downarrow$$

$$\boxed{(-1)^{2m_2}} (-1)^{j_2 - m_2} \delta_{j_1 j_2} \delta_{m_1 - m_2}$$

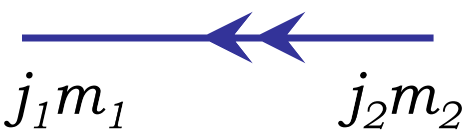
$$(-1)^{2m_2} = (-1)^{2m_2 + 2(j_2 - m_2)} = (-1)^{2j_2}$$

Two arrows on the line

Same direction: extra phase factor

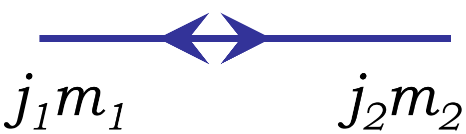


$= (-1)^{2j_2}$



$= (-1)^{2j_2}$

Opposite direction arrows cancel out



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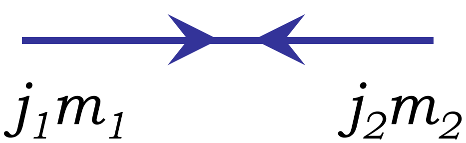

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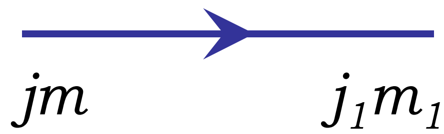
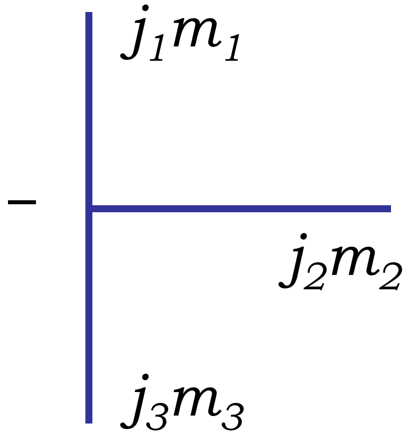
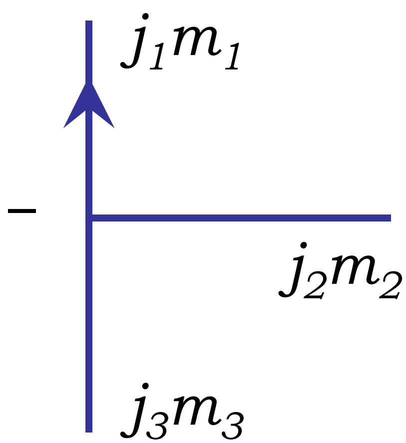


$= (-1)^{j_1 - m_1}$

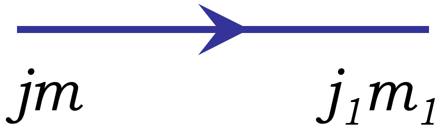
Arrows on 3-j symbols



$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

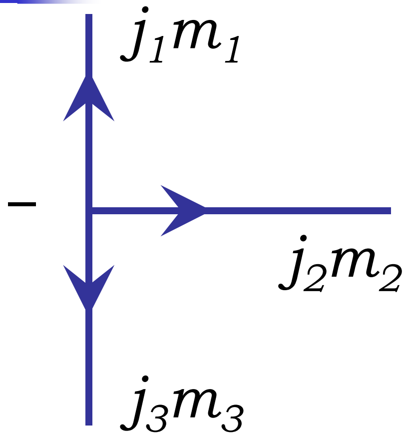


$$= (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{pmatrix}$$



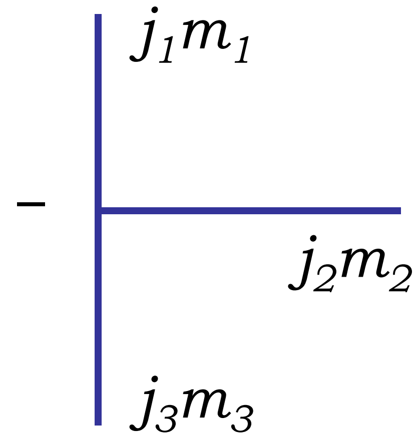
$= (-1)^{j_1-m_1}$

Arrows & vertexes



$$= (-1)^{(j_1-m_1)+(j_2-m_2)+(j_3-m_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

$$= (-1)^{2(j_1+j_2+j_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$$

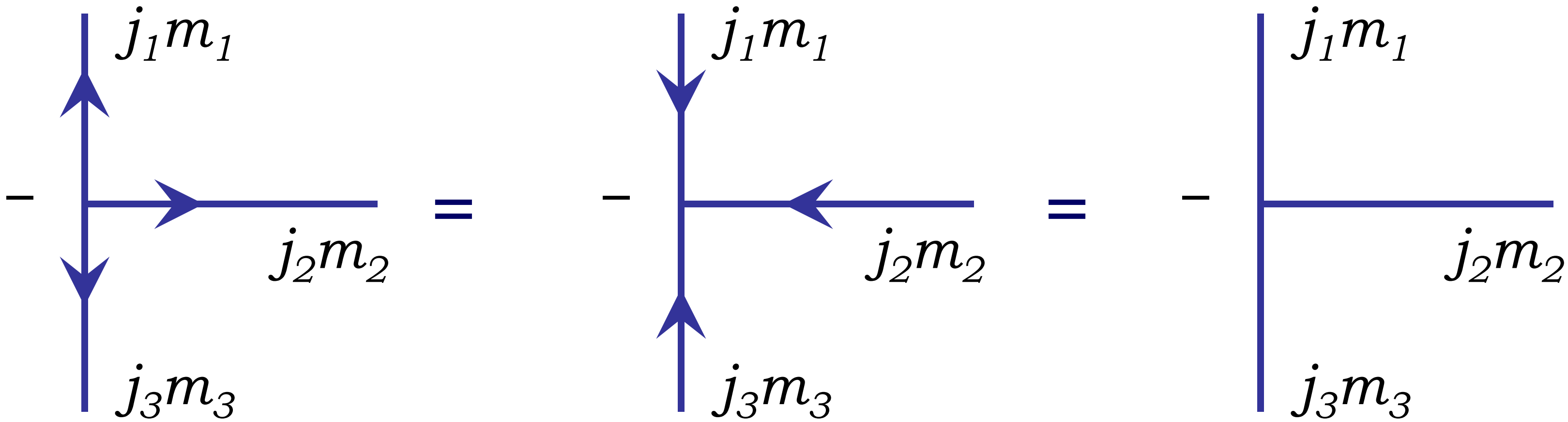


$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$m_1 + m_2 + m_3 = 0$$

Arrows & vertexes

One can put three same direction arrows on the vertex
(all three must be directed to the vertex or out of the vertex)

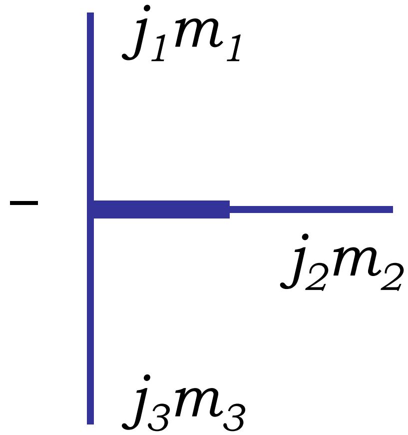


$$[j] \equiv 2j+1$$

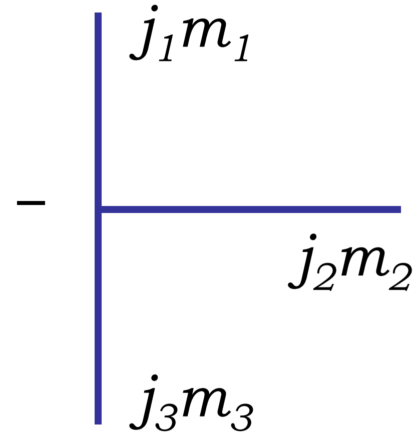
$\sqrt{2j+1}$ factor

$\sqrt{2j+1}$ factor is represented by thickening part of the line

$$= \sqrt{2j_1+1}$$



$$= \sqrt{2j_2+1}$$



How to draw a Clebsch-Gordon coefficient ?

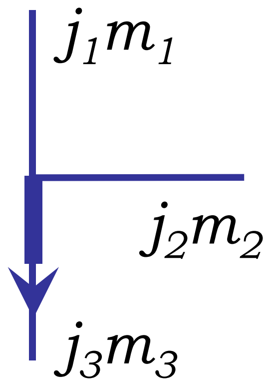
$$\overbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}^{3-j\ symbol} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \overbrace{\left\langle j_1 m_1 j_2 m_2 \middle| j_3 -m_3 \right\rangle}^{Clebsch-Gordan\ coefficient}$$

$$\left\langle j_1 m_1 j_2 m_2 \middle| j_3 m_3 \right\rangle = (-1)^{j_1-j_2+m_3} \sqrt{2j_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

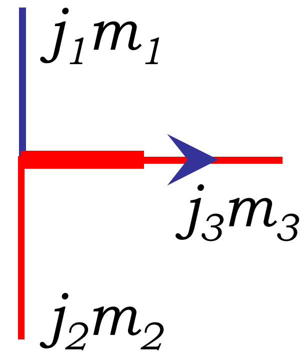
How to draw a Clebsch-Gordon coefficient ?

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 - j_2 + m_3} \sqrt{2 j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

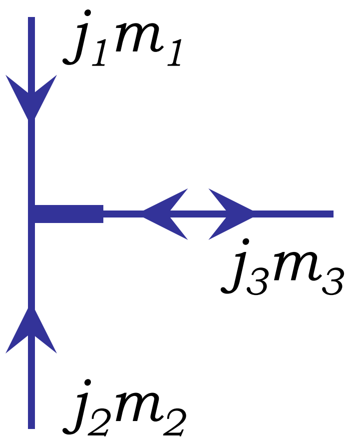
$$= (-1)^{j_1 - j_2 + m_3 + (j_3 - m_3)} \; -$$



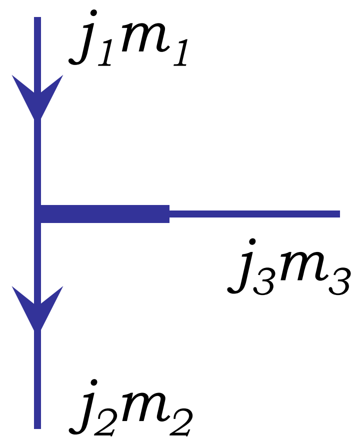
$$= (-1)^{j_1 - j_2 + j_3 + (j_1 + j_2 + j_3)} \; -$$



$$= (-1)^{2 j_2} \; -$$

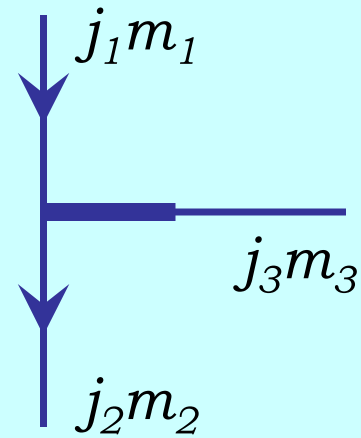


$$= \; -$$



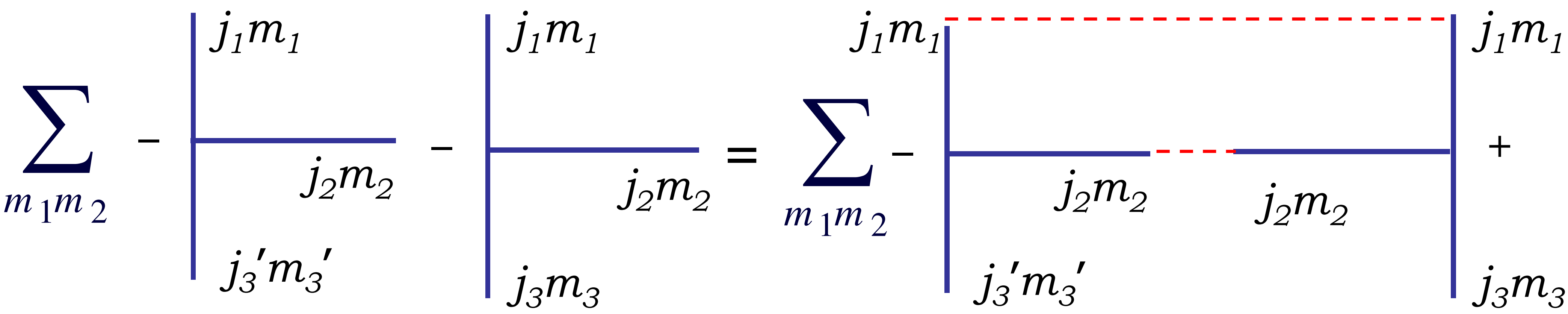
How to draw a Clebsch-Gordon coefficient ?

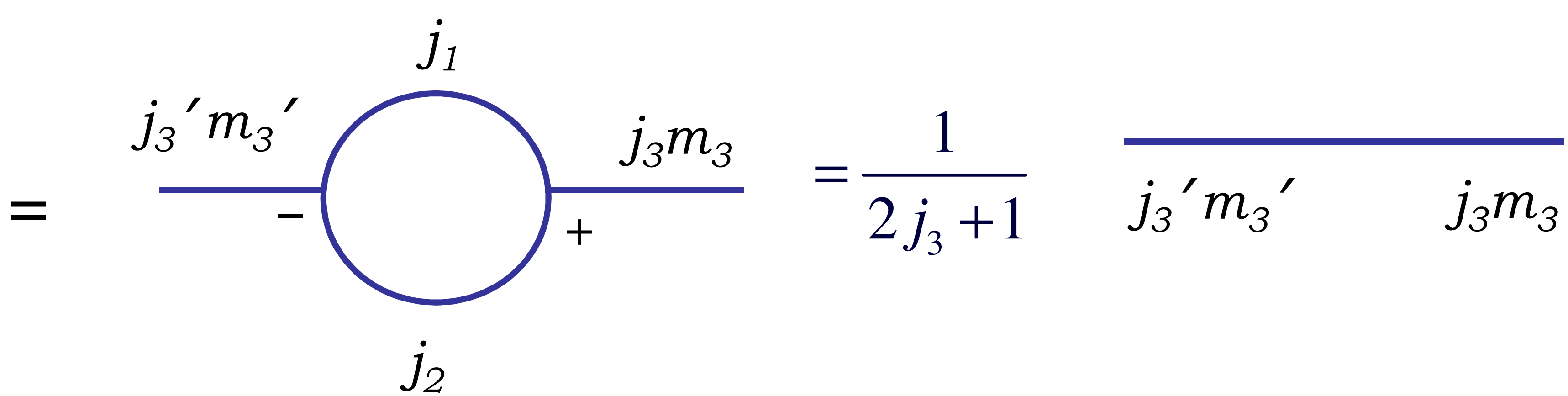
$$\langle j_1 m_1 j_2 m_2 \mid j_3 m_3 \rangle = -$$



Putting it all together: orthogonality relation

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{1}{2j_3+1} \delta_{j_3' j_3} \delta_{m_3' m_3}$$





Irreducible tensor operators

Irreducible tensor operators: a family of $2k+1$ operators with $q=-k,-k+1,\ \ldots,\ k$ satisfying the commutation relations

$$\left[J_z, T_q^k \right] = q T_q^k$$

$$\left[J_\pm, T_q^k \right] = \sqrt{(k \pm q + 1)(k \mp q)} T_{q\pm1}^k$$

How to calculate their matrix elements?

Wigner-Eckart theorem

Matrix elements of irreducible tensor operators between angular momentum states are evaluated using **Wigner-Eckart theorem**

$$\langle j_1 m_1 | T_q^k | j_2 m_2 \rangle = (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix} \langle j_1 \| T^k \| j_2 \rangle$$

Reduced matrix elements: no dependence on the magnetic quantum numbers

$$\langle j_1 m_1 | T_q^k | j_2 m_2 \rangle \neq 0 \text{ if}$$

$$q = m_1 - m_2$$

$$|j_1 - j_2| \leq k \leq j_1 + j_2$$

Transition selection rules

Irreducible tensor operators: examples

Spherical harmonics $Y_{lm}(\theta,\phi)$ are irreducible tensor operators of rank l.

Operators Jμ are irreducible tensor operators of rank 1.

$$J_q = \begin{cases} -\frac{1}{\sqrt{2}}\left(J_x + iJ_y\right) & q=1 \\ J_z & q=0 \\ \frac{1}{\sqrt{2}}\left(J_x - iJ_y\right) & q=-1 \end{cases}$$

$$\left\langle j_1 m_1 \middle| J_q \middle| j_2 m_2 \right\rangle = (-1)^{j_1-m_1} \begin{pmatrix} j_1 & 1 & j_2 \\ -m_1 & q & m_2 \end{pmatrix} \left\langle j_1 \| J \| j_2 \right\rangle$$

Irreducible tensor operators: examples
How to evaluate $\langle j_1 \| J \| j_2 \rangle$

$$\langle j_1 m_1 | J_q | j_2 m_2 \rangle = (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & 1 & j_2 \\ -m_1 & q & m_2 \end{pmatrix} \langle j_1 \| J \| j_2 \rangle$$

Consider case with q=0, $J_0=J_z$

$$m_1 \delta_{j_1 j_2} \delta_{m_1 m_2} = (-1)^{j_1 - m_1} \underbrace{\begin{pmatrix} j_1 & 1 & j_1 \\ -m_1 & 0 & m_1 \end{pmatrix}}_{(-1)^{j_1 - m_1} \frac{m_1}{\sqrt{j_1(j_1+1)(2j_1+1)}}} \langle j_1 \| J \| j_2 \rangle$$

$$\langle j_1 \| J \| j_2 \rangle = \sqrt{j_1(j_1+1)(2j_1+1)}\, \delta_{j_1 j_2}$$

Irreducible tensor operators
Dipole operator **d**=-e**r**

$$r_q = \begin{cases} -\frac{1}{\sqrt{2}}(x+iy) = -\frac{r}{\sqrt{2}}(\sin\theta\cos\phi + i\sin\theta\sin\phi) = -\frac{r}{\sqrt{2}}\sin\theta e^{i\phi} & q=1 \\ z = r\cos\theta & q=0 \\ \frac{1}{\sqrt{2}}(x-iy) = \frac{r}{\sqrt{2}}(\sin\theta\cos\phi - i\sin\theta\sin\phi) = -\frac{r}{\sqrt{2}}\sin\theta e^{-i\phi} & q=-1 \end{cases}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

$$r_q = \begin{cases} r\sqrt{\frac{4\pi}{3}}Y_{11} & q=+1 \\ r\sqrt{\frac{4\pi}{3}}Y_{10} & q=0 \\ r\sqrt{\frac{4\pi}{3}}Y_{1-1} & q=-1 \end{cases}$$

$$C_q^k = \sqrt{\frac{4\pi}{(2k+1)}}Y_{kq}(\theta,\phi)$$

Quadrupole operator:

$$Q_q^{(2)} = r^2 C_q^{(2)}(\hat{r})$$

Irreducible tensor operators
Coulomb interaction

$$C_q^k(\hat{r}) = \sqrt{\frac{4\pi}{(2k+1)}} Y_{kq}(\theta,\phi)$$

$1/\mathbf{r}_{12}$ can be expanded in partial waves

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{k=0}^{\infty} \frac{r_<^k}{r_>^{k+1}} P_k(\cos\theta)$$

Legendre polynomial

$r_<$ and $r_>$ are lesser and greater of r_1, r_2
θ is the angle between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$

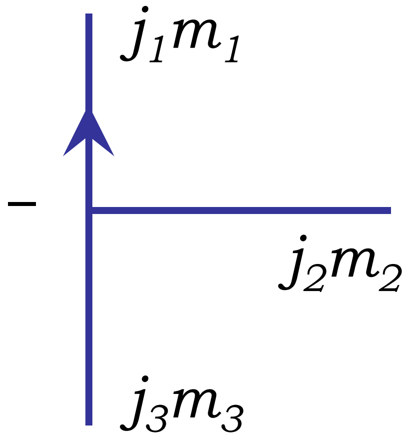
$$P_k(\cos\theta) = \frac{4\pi}{2k+1} \sum_{q=-k}^{k} Y_{kq}(\theta_1,\phi_1) Y_{kq}^*(\theta_2,\phi_2)$$

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{k=0}^{\infty} \frac{r_<^k}{r_>^{k+1}} \sum_{q=-k}^{k} (-1)^q\, C_q^k(\hat{r}_1)\, C_{-q}^k(\hat{r}_2)$$

Wigner-Eckart theorem: graphical form

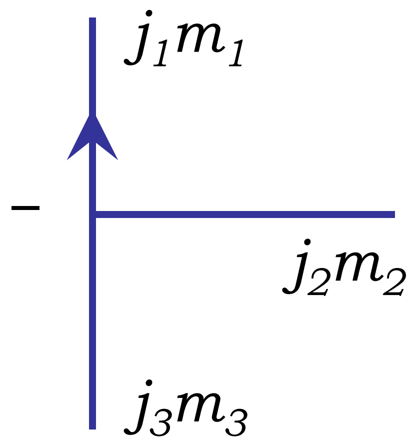
$$\langle j_1 m_1 | T^k_q | j_2 m_2 \rangle = (-1)^{j_1-m_1} \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix} \langle j_1 \| T^k \| j_2 \rangle$$

Use



$$= (-1)^{j_1-m_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{pmatrix}$$

$$\langle j_1 m_1 | T^k_q | j_2 m_2 \rangle =$$

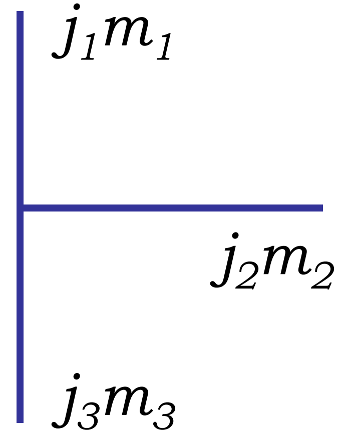


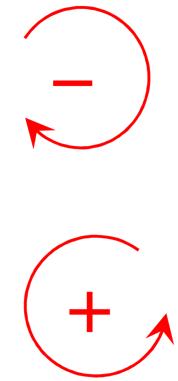
$$\langle j_1 \| T^k \| j_2 \rangle$$

Summary of graphical representation: three and half basic concepts

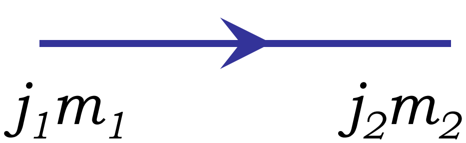
Vertex: 3-j symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \quad -$$





Directed line



$= (-1)^{j_2-m_2}$

$= (-1)^{j_2-m_2}\,\delta_{j_1 j_2}\,\delta_{m_1 -m_2}$

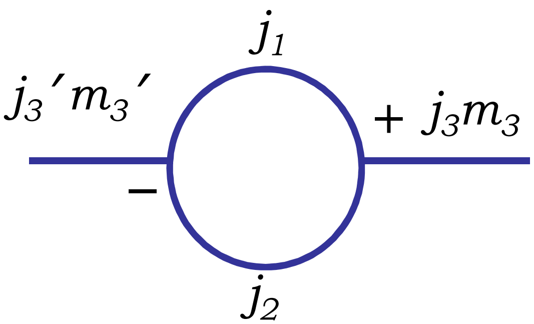
Sum over m: join the corresponding lines

Thick line represents $\sqrt{2j+1}$ factor

Derivative rules, symmetries, orthogonality relation & commonly used constructs

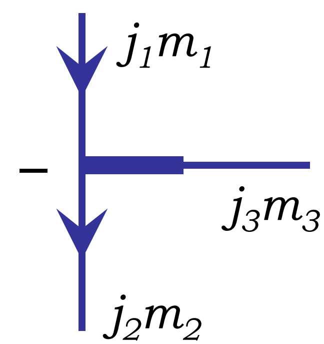
Arrows:
- two opposite direction arrows cancel
- two same direction arrows give $(-1)^{2j}$ factor
- switching direction of an arrow gives $(-1)^{2j}$ factor

Vertexes:
- switching between "−" and "+" gives $(-1)^{j1+j2+j3}$ factor
- switching of two lines gives $(-1)^{j1+j2+j3}$ factor
- one can add three same direction arrows on a vertex



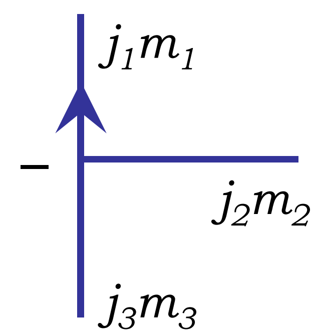
$$= \frac{1}{2j_3+1}\delta_{j_3' j_3}\delta_{m_3' m_3}$$

Orthogonality relation



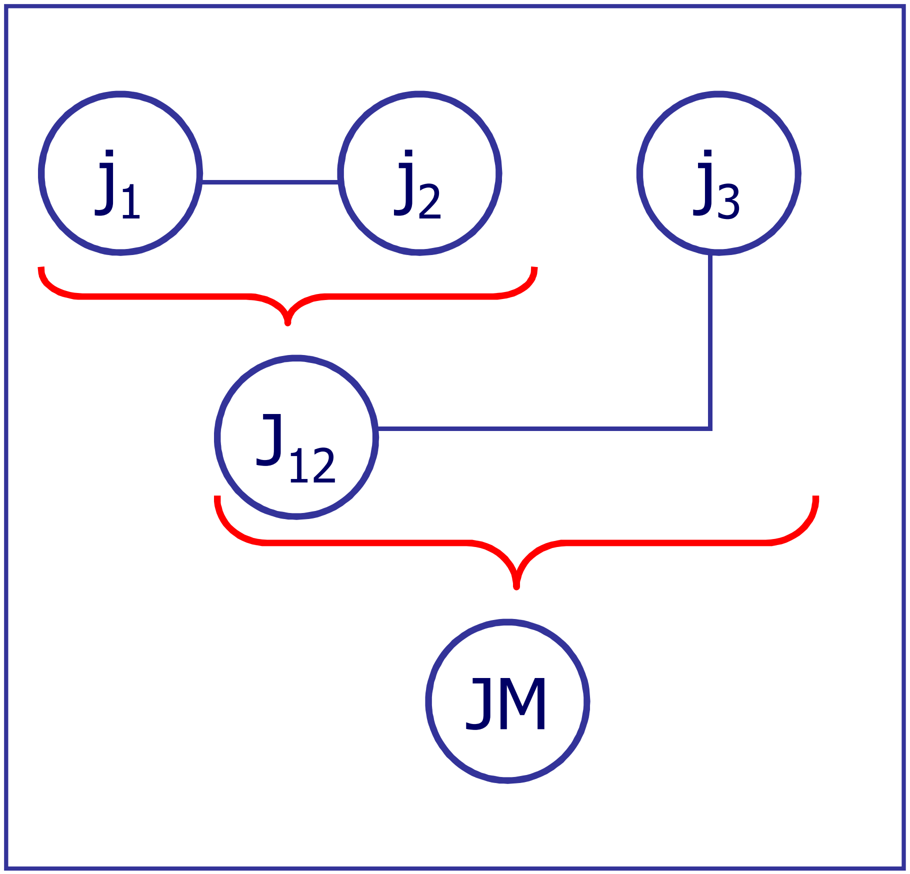
Clebsch-Gordon coefficient

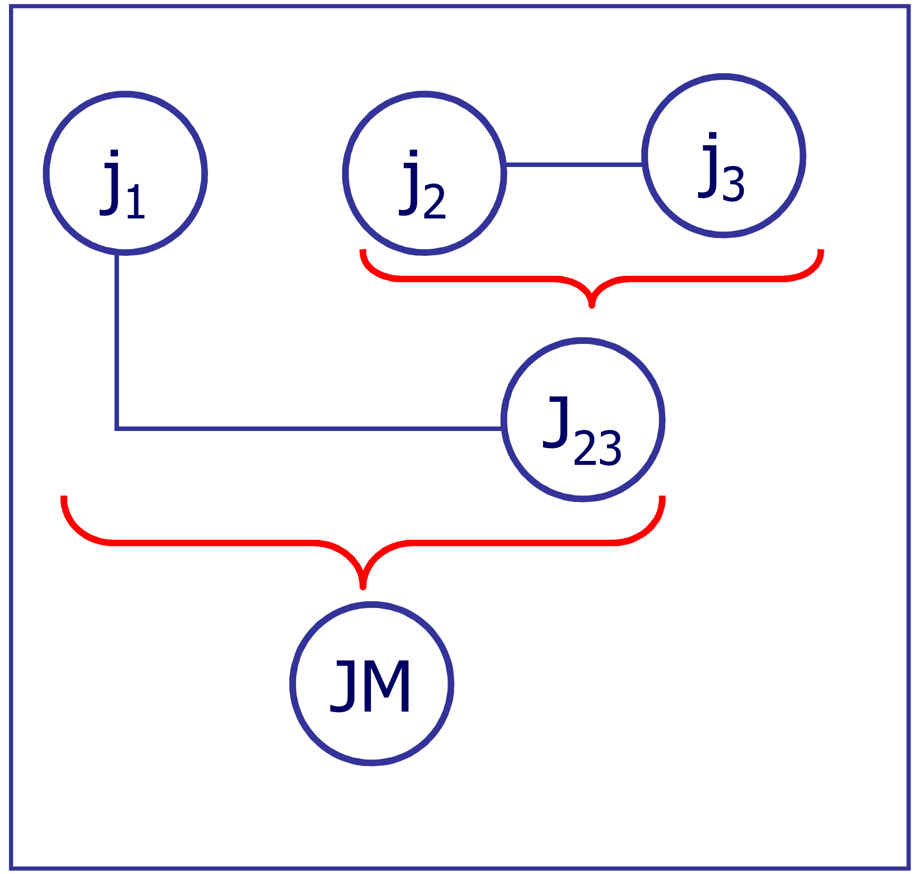
$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$



Symbol in Wigner-Eckart theorem

Three states & 6-j symbol

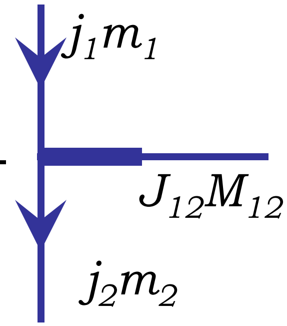




$$\left|(j_1 j_2)J_{12}\,j_3,JM\right\rangle \qquad \left|j_1(j_2 j_3)J_{23},JM\right\rangle$$

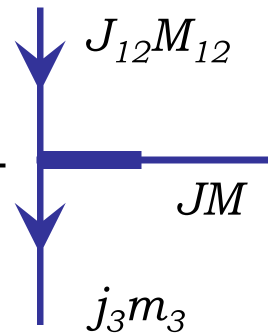
Three states & 6-j symbol

$$\left|(j_1 j_2) J_{12} M_{12}\right\rangle = \sum_{m_1 m_2} -$$



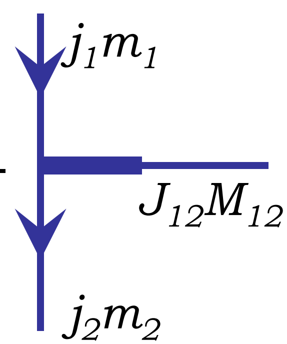
$$\left|j_1 m_1\right\rangle \left|j_2 m_2\right\rangle$$

$$\left|(J_{12} j_3) JM\right\rangle = \sum_{M_{12} m_3} -$$

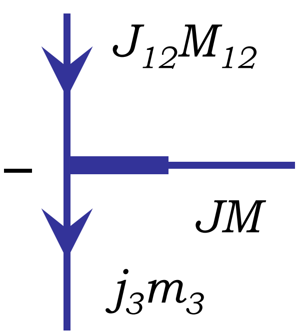


$$\left|(j_1 j_2) J_{12} M_{12}\right\rangle \left|j_3 m_3\right\rangle$$

$$\left|(j_1 j_2) J_{12} j_3, JM\right\rangle = \sum_{\substack{m_1 m_2 m_3 \\ M_{12}}} -$$



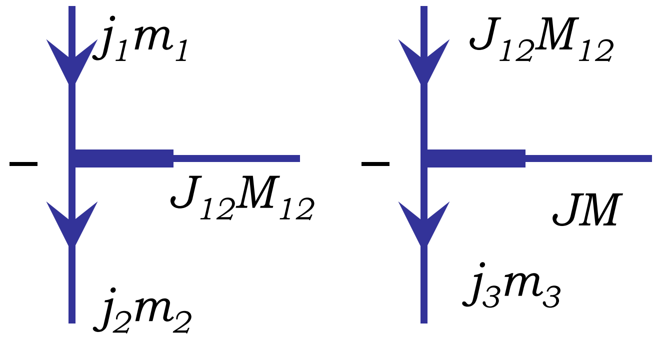
$$-$$



$$\left|j_1 m_1\right\rangle \left|j_2 m_2\right\rangle \left|j_3 m_3\right\rangle$$

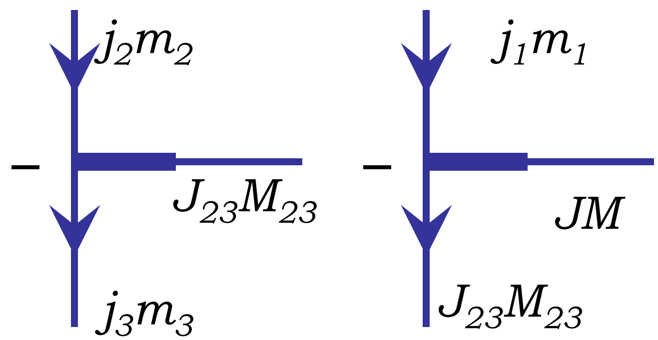
Three states & 6-j symbol

$$\left|(j_1 j_2)J_{12} j_3, JM\right\rangle = \sum_{\substack{m_1 m_2 m_3 \\ M_{12}}}$$



$$\left|j_1 m_1\right\rangle \left|j_2 m_2\right\rangle \left|j_3 m_3\right\rangle$$

$$\left|(j_1 (j_2 j_3)J_{23}, JM\right\rangle = \sum_{\substack{m_1 m_2 m_3 \\ M_{12}}}$$



$$\left|j_1 m_1\right\rangle \left|j_2 m_2\right\rangle \left|j_3 m_3\right\rangle$$

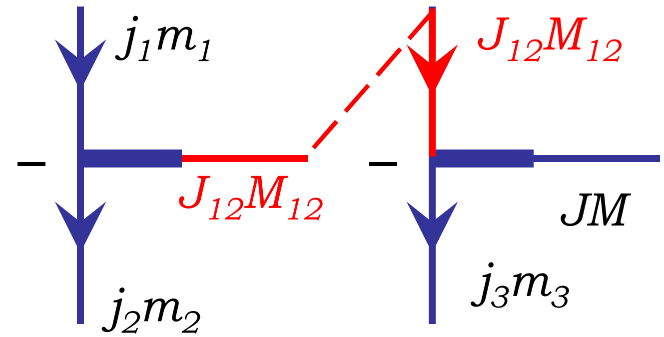
Express states from 1(23) scheme as linear combination of the states obtained with (12)3 scheme

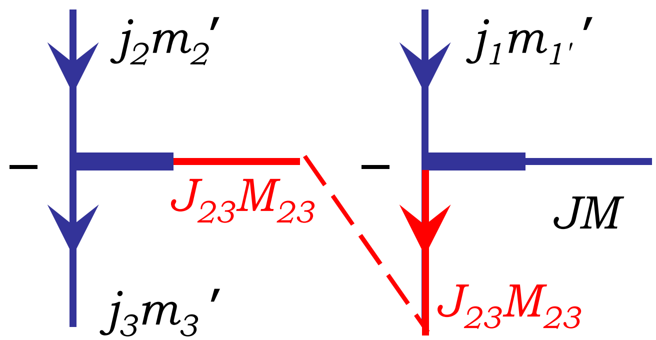
$$\left|(j_1 (j_2 j_3)J_{23}, JM\right\rangle = \sum_{J_{12}} \left|(j_1 j_2)J_{12} j_3, JM\right\rangle \left\langle (j_1 j_2)J_{12} j_3, JM \middle| (j_1 (j_2 j_3)J_{23}, JM\right\rangle$$

Coupling coefficient does not depend on M

Three states & 6-j symbol

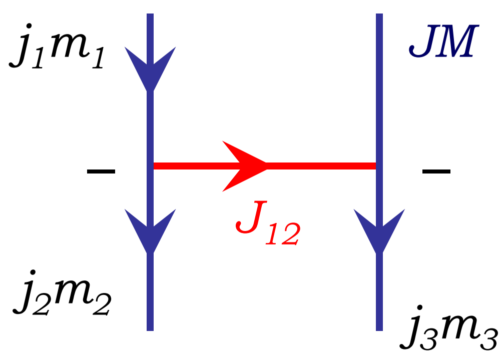
$$\langle (j_1 j_2)J_{12} j_3, JM \,|\, (j_1(j_2 j_3)J_{23}, JM\rangle = \sum_{\substack{m_1 m_2 m_3 \\ m_1' m_2' m_3' \\ M_{12} M_{23}}}$$

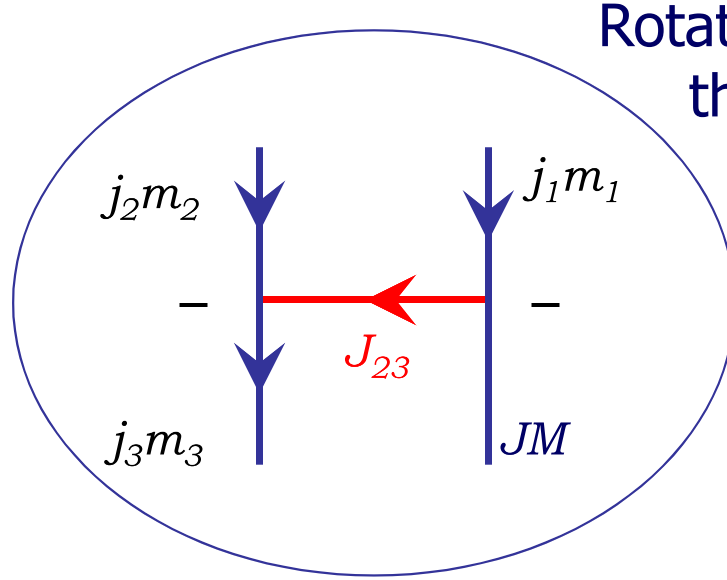




$$\langle j_1 m_1' | j_1 m_1\rangle \langle j_2 m_2' | j_2 m_2\rangle \langle j_3 m_3' | j_3 m_3\rangle$$

$$= \sum_{m_1 m_2 m_3} [J]\sqrt{[J_{12}][J_{23}]}$$

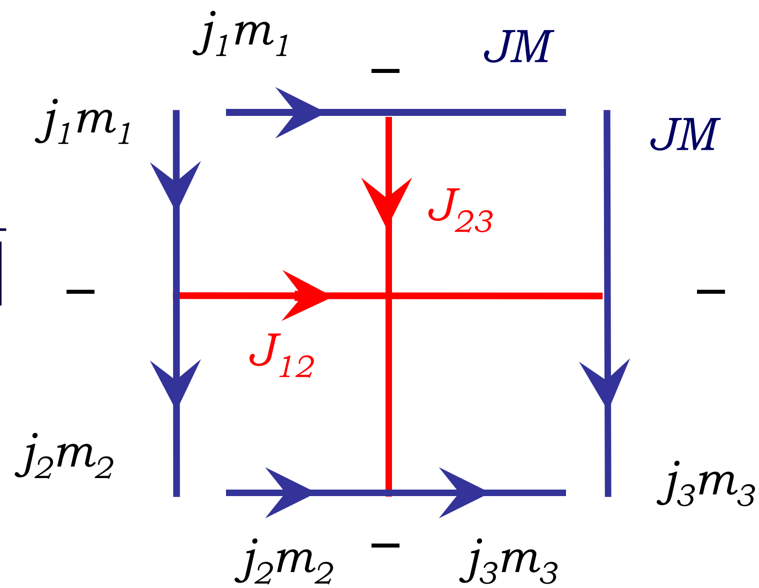




Rotate 90° to the left

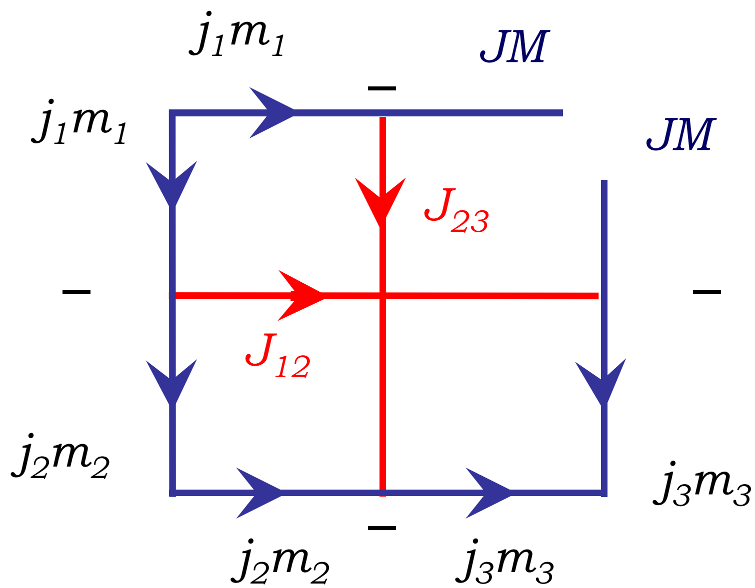
$$\langle (j_1 j_2)J_{12} j_3, JM \,|\, (j_1 (j_2 j_3)J_{23}, JM \rangle$$

$$= \sum_{m_1 m_2 m_3} [J]\sqrt{[J_{12}][J_{23}]}$$



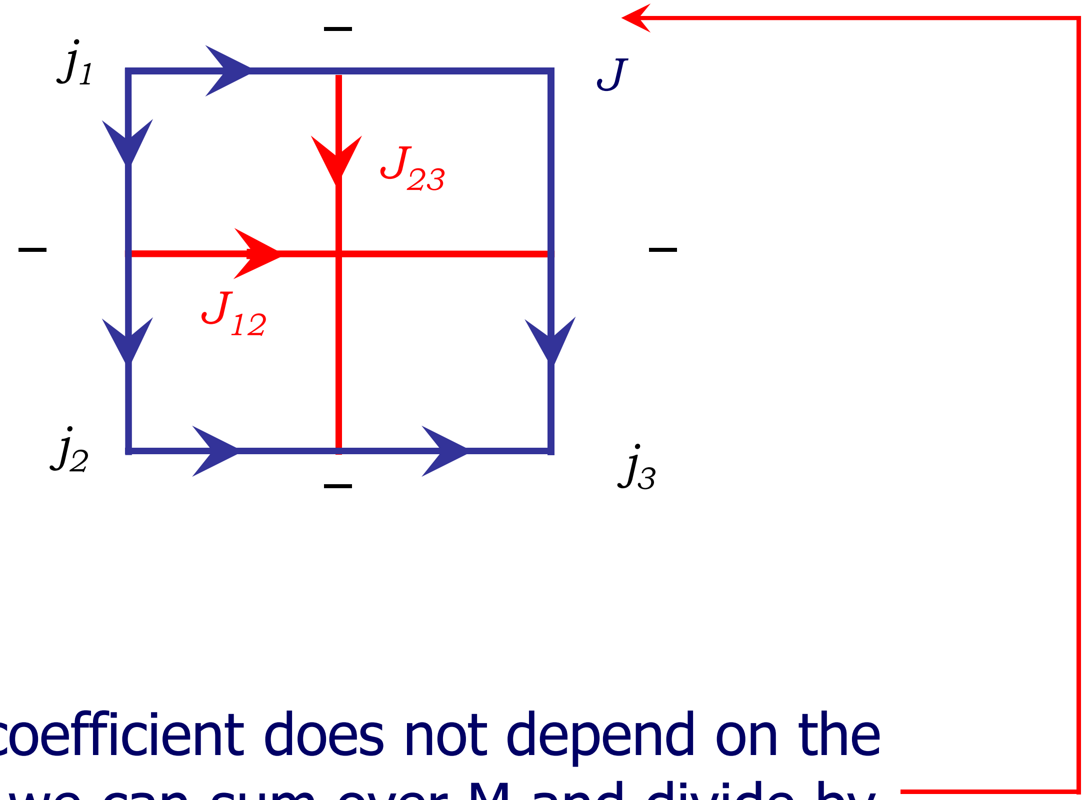
$$\langle (j_1 j_2)J_{12} j_3, JM \,|\, (j_1 (j_2 j_3)J_{23}, JM \rangle$$

$$= [J]\sqrt{[J_{12}][J_{23}]}$$



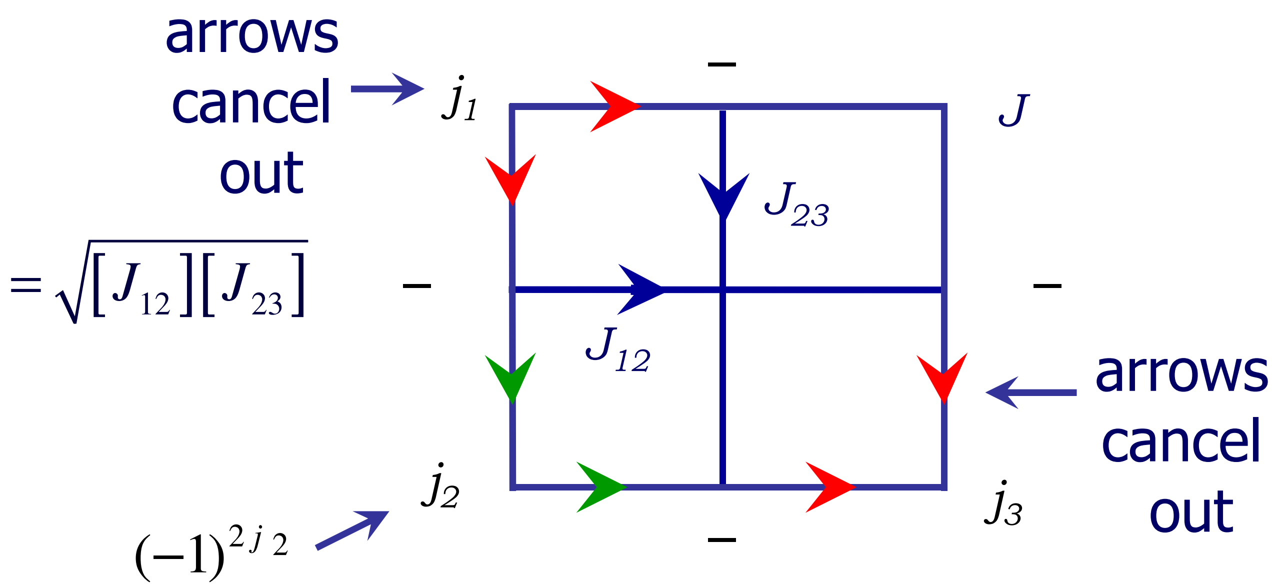
$$\left\langle (j_1 j_2)J_{12} j_3, JM \,\middle|\, (j_1(j_2 j_3)J_{23}, JM \right\rangle$$

$$= \sqrt{[J_{12}][J_{23}]}$$



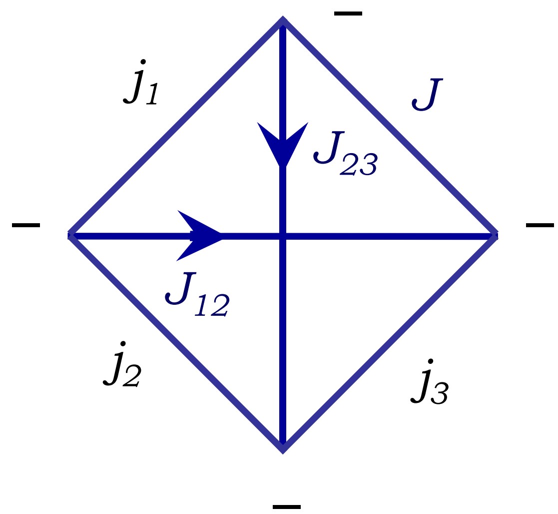
This coupling coefficient does not depend on the value of M, so we can sum over M and divide by the number of different M states, i.e. (2J+1)

$$\left\langle (j_1 j_2) J_{12} j_3, JM \middle| (j_1 (j_2 j_3) J_{23}, JM \right\rangle$$

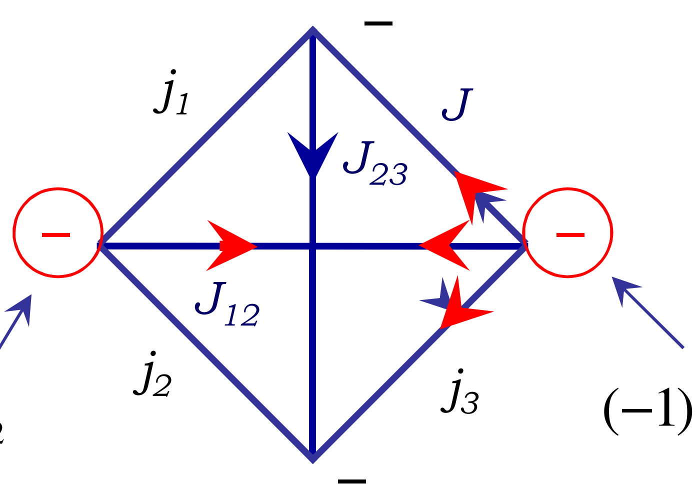


$$\left\langle (j_1 j_2) J_{12} j_3, JM \middle| (j_1 (j_2 j_3) J_{23}, JM \right\rangle$$

$$= (-1)^{2j_2} \sqrt{[J_{12}][J_{23}]}$$



$$= (-1)^{2j_2} \sqrt{[J_{12}][J_{23}]}$$

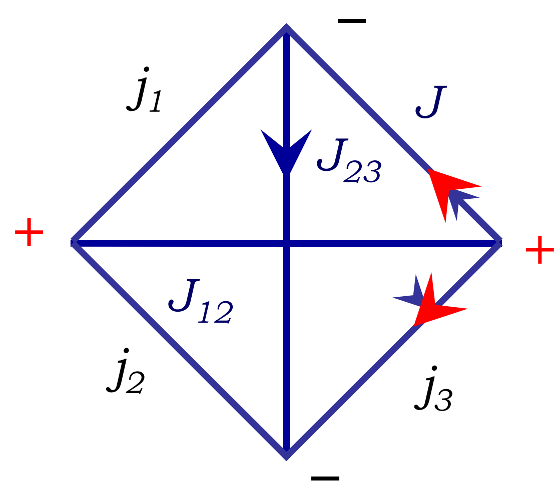


$(-1)^{j_1+j_2+J_{12}}$

$(-1)^{J+j_3+J_{12}}$

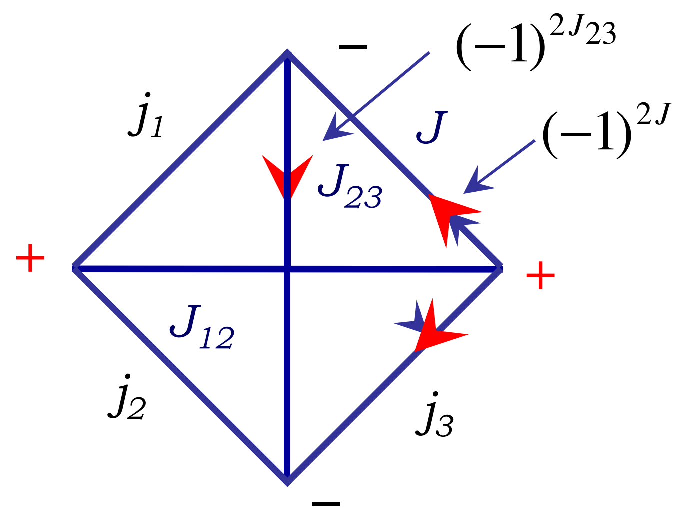
$$\left\langle (j_1 j_2)J_{12} j_3, JM \middle| (j_1(j_2 j_3)J_{23}, JM \right\rangle$$

$$= (-1)^{2j_2}\sqrt{[J_{12}][J_{23}]}$$



$$(-1)^{j_1+j_2+J_{12}+J+j_3+J_{12}}$$

$$= (-1)^{2j_2}\sqrt{[J_{12}][J_{23}]}$$

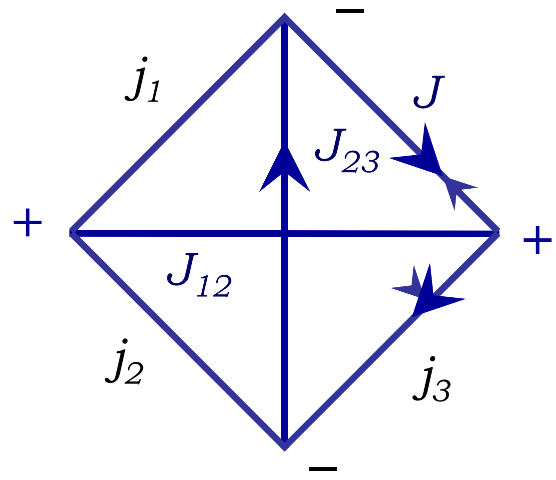


$$(-1)^{j_1+j_2+J+j_3}$$

Three states & 6-j symbol

$$\langle (j_1 j_2) J_{12} j_3, JM | (j_1 (j_2 j_3) J_{23}, JM \rangle$$

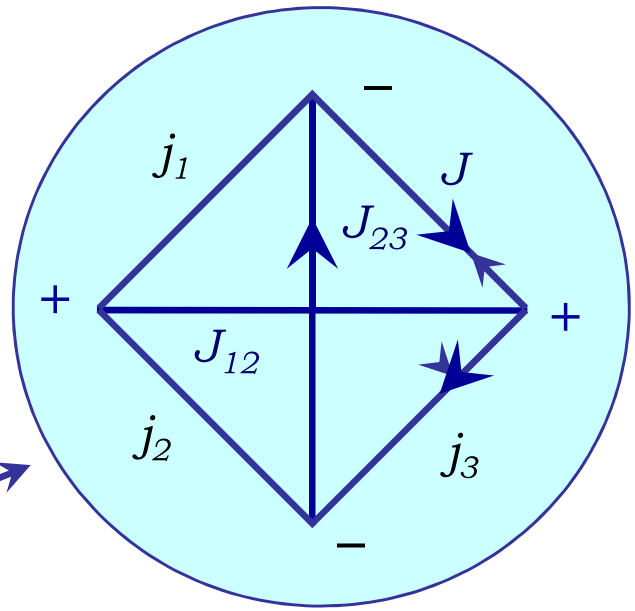
$$= \sqrt{[J_{12}][J_{23}]} \quad (-1)^{j_1 + j_2 + J + j_3}$$



Three states & 6-j symbol

$$\langle (j_1 j_2) J_{12} j_3, JM \,|\, (j_1 (j_2 j_3) J_{23}, JM \rangle$$

$$= \sqrt{[J_{12}][J_{23}]}$$



$$(-1)^{j_1+j_2+J+j_3}$$

6-j symbol

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix}$$

Triangular conditions for 6-j symbol

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix}$$

Triangular conditions:

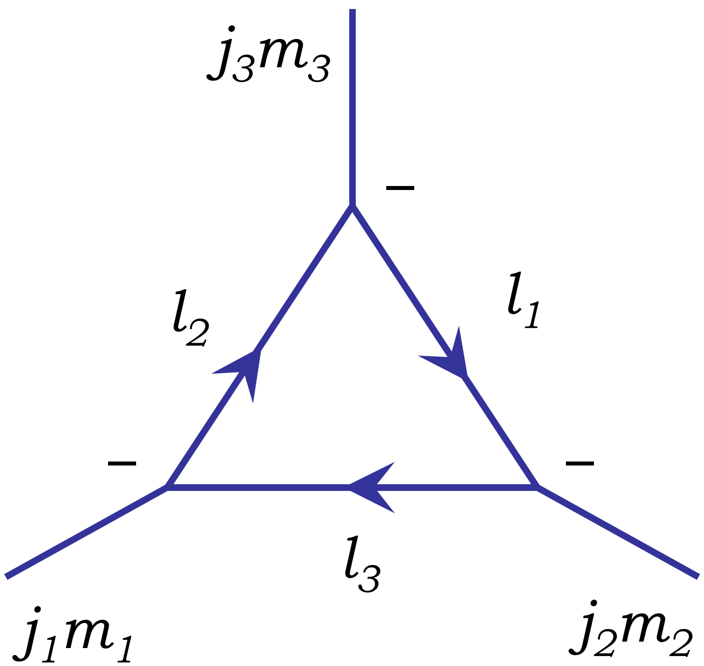
$(j_1 j_2 J_{12})$ $(j_3 J J_{12})$ $(j_3 j_2 J_{23})$ $(j_1 J_{23} J)$

$$\begin{Bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \cdot & \cdot & \cdot \end{Bmatrix} \quad \begin{Bmatrix} \cdot & \cdot & \bigcirc \\ \bigcirc & \bigcirc & \cdot \end{Bmatrix} \quad \begin{Bmatrix} \cdot & \bigcirc & \cdot \\ \bigcirc & \cdot & \bigcirc \end{Bmatrix} \quad \begin{Bmatrix} \bigcirc & \cdot & \cdot \\ \cdot & \bigcirc & \bigcirc \end{Bmatrix}$$

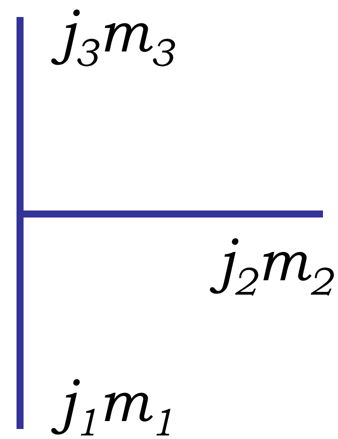
$$\left| j_1 - j_2 \right| \leq J_{12} \leq j_1 + j_2$$

← Triangular condition

6-j symbol & how to remove triangle from the graph



$$= \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} +$$



6-j symbol symmetry

6-j symbol is invariant with respect to any permutations of columns

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ l_2 & l_1 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_1 \\ l_2 & l_3 & l_1 \end{Bmatrix}$$

6-j symbol is invariant under the inversion of the arguments in any two columns

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_1 & l_2 & l_3 \\ l_1 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} l_1 & j_2 & l_3 \\ j_1 & l_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} l_1 & l_2 & j_3 \\ j_1 & j_2 & l_3 \end{Bmatrix}$$