Lecture #21

A particle in electromagnetic field

Aharonov-Bohm effect

Flux quantization in superconductors

Superconducting devices

Jasprit Singh, Quantum Mechanics, pages 435-439

Bransden & Joachain, Quantum Mechanics, Chapter 15, pages 567-571

A particle in electromagnetic field

We consider a particle of mass m and charge q moving in electromagnetic field described by a vector potential **A**(**r**,t) and a scalar potential ϕ(**r**,t).
We can obtain non-relativistic classical Hamiltonian by taking

$$E = \frac{\mathbf{p}^2}{2m}$$ and making replacements $$E \to E - q\phi$$ $$\mathbf{p} \to \mathbf{p} - q\mathbf{A}$$

$$H = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 + q\phi.$$

In quantum mechanics, we use correspondence principle:
to obtain

$$E \to E_{op} = i\hbar\frac{\partial}{\partial t}$$

$$\mathbf{p} \to \mathbf{p}_{op} = -i\hbar\nabla$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar}{2m}\left[\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\right] + \frac{q^2}{2m}\mathbf{A}^2 + q\phi\right\}\psi.$$

The questions

We investigate the following questions:

The charged particle of mass m and charge q moves only in regions of vanishing **B(r)** but nonvanishing **A**(**r**).

1. Does it feel the existence of the magnetic field in the inaccessible region?

2. Can it be experimentally verified?

B=0

$$\mathbf{B} = \nabla \times \mathbf{A} = 0$$

$Case\,1.\quad \mathbf{A} = 0$

$Case\,2.\quad \mathbf{A} = \nabla \chi$

Some scalar function

$$Case\,1.\quad \left\{ -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) \right\} \psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$Case\,2.\quad \left\{ -\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar}{2m}\left[\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\right] + \frac{q^2}{2m}\mathbf{A}^2 + V(\mathbf{r}) \right\} \psi'(\mathbf{r}) = E\psi'(\mathbf{r})$$

We assume **A** to be time independent.

$\psi(\mathbf{r})$ vs. $\psi'(\mathbf{r})$

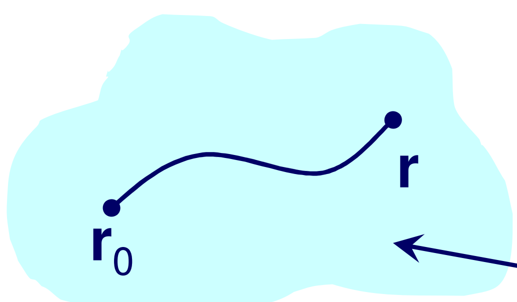
$$Case\,1.\ \left\{-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})\right\}\psi(\mathbf{r})=E\psi(\mathbf{r})$$

$$Case\,2.\ \left\{-\frac{\hbar^2}{2m}\nabla^2+\frac{i\hbar}{2m}[\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla]+\frac{q^2}{2m}\mathbf{A}^2+V(\mathbf{r})\right\}\psi'(\mathbf{r})=E\psi'(\mathbf{r})$$

$$\psi'(\mathbf{r})=\psi(\mathbf{r})e^{iq\chi(\mathbf{r})/\hbar}$$

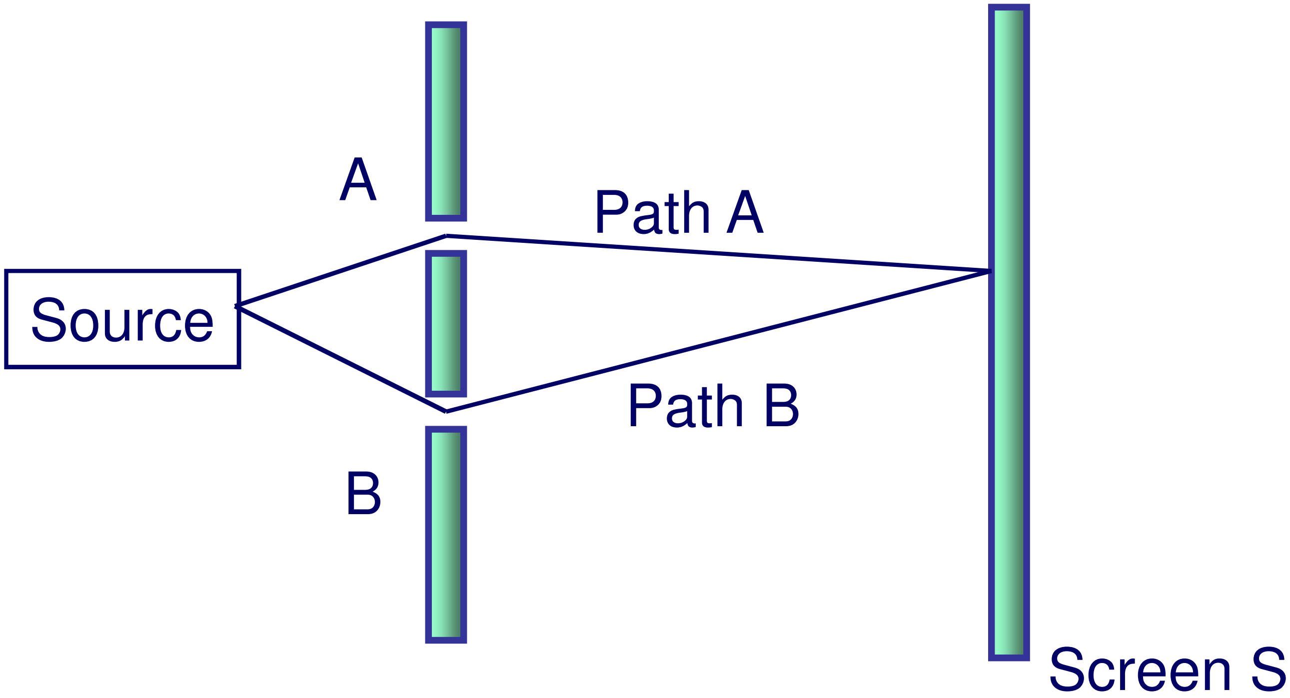
$$\mathbf{A}(\mathbf{r})=\nabla\chi(\mathbf{r})\ \rightarrow\ \chi(\mathbf{r})=\int_{\mathrm{r}_0}^{\mathrm{r}}\mathbf{A}(\mathbf{r}')\,\mathbf{dr}'$$

Line integral from some fixed point $\mathbf{r}_0$ to $\mathbf{r}$.

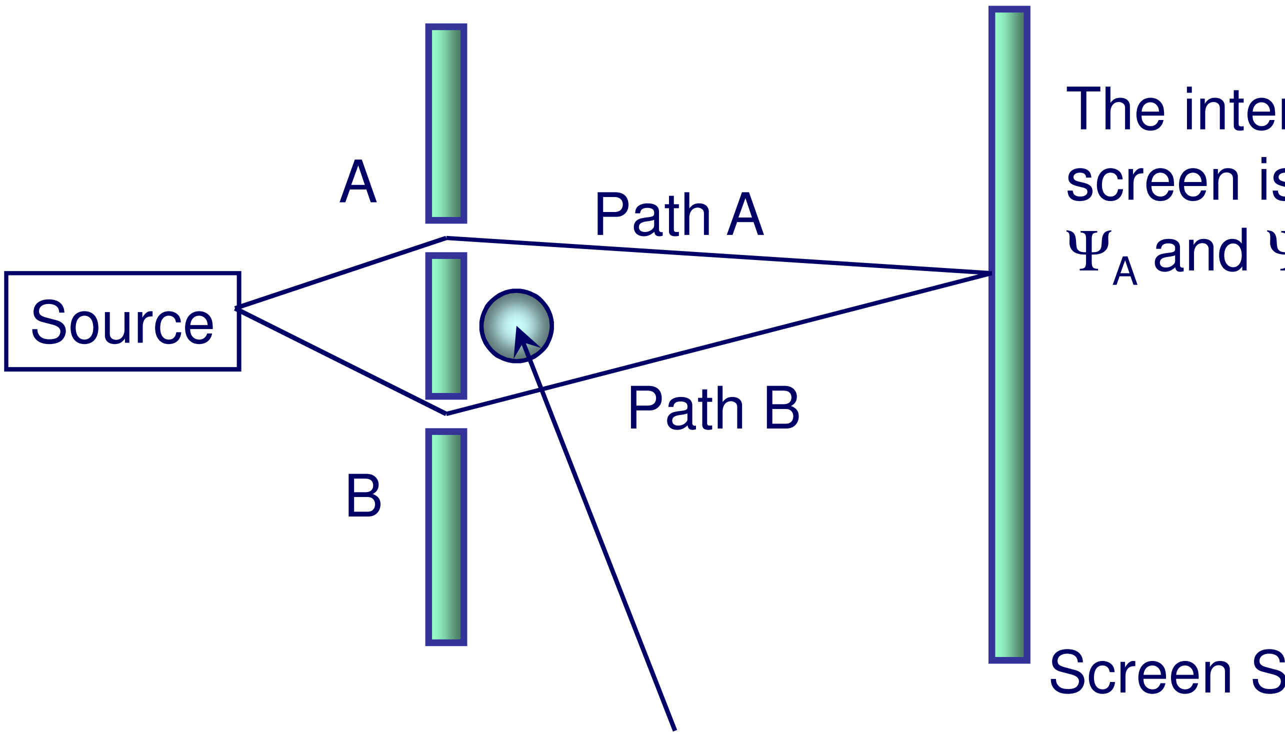


B=0 region

The interference experiment



The interference experiment

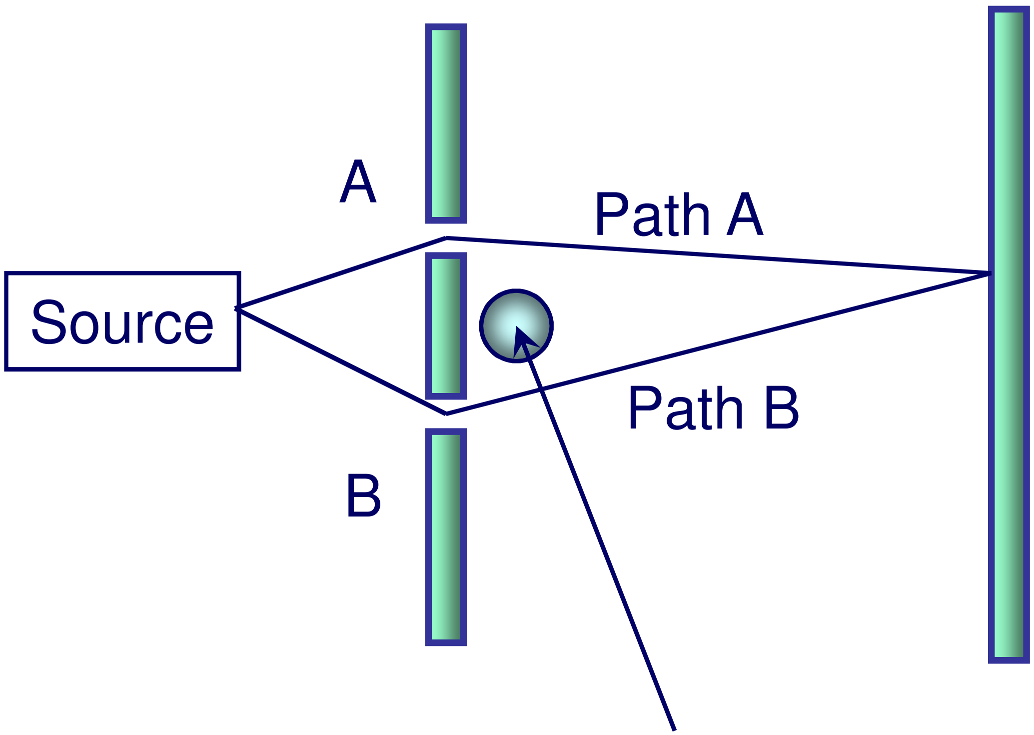


The interference pattern on the screen is due to superposition of Ψ_A and Ψ_B.

A thin solenoid (perpendicular to the screen), which contains a magnetic flux Φ.

1. There is no magnetic field outside the solenoid.
2. Solenoid is impenetrable to the charge particles. therefore, our particles always moves in B=0 region

The interference experiment



Screen S

The interference pattern on the screen is due to superposition of Ψ_A and Ψ_B.

When $\Phi = 0$

$$\Psi^{(0)} = \Psi_A^{(0)} + \Psi_B^{(0)}$$

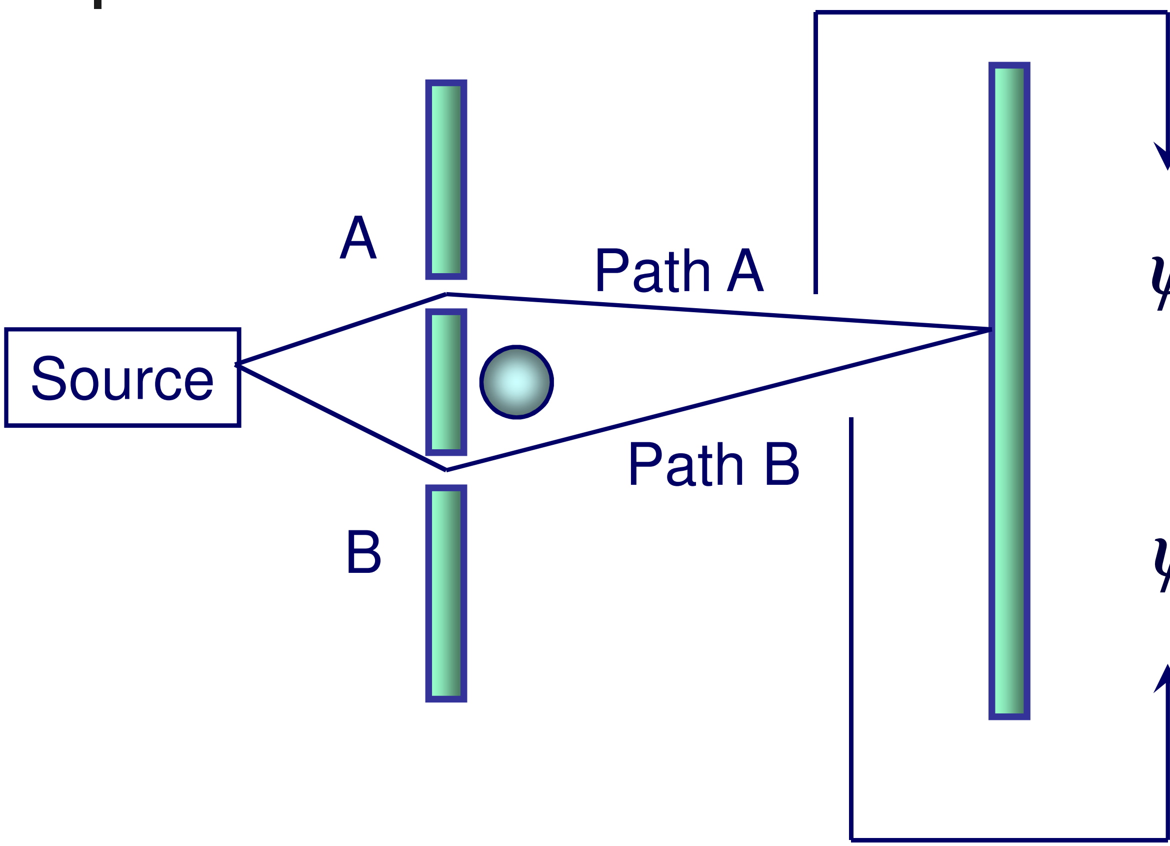
(Corresponds to Case 1: $\mathbf{A} = 0$)

A thin solenoid (perpendicular to the screen), which contains a magnetic flux Φ.

1. There is no magnetic field outside the solenoid.
2. Solenoid is impenetrable to the charge particles. therefore, our particles always move in B=0 region

The interference experiment

Case 2: $\Phi \neq 0$



$$\psi_A = \psi_A^{(0)}\, e^{\frac{iq}{\hbar}\int_{path\ \mathrm{A}} \mathbf{A}(\mathbf{r})d\mathbf{r}}$$

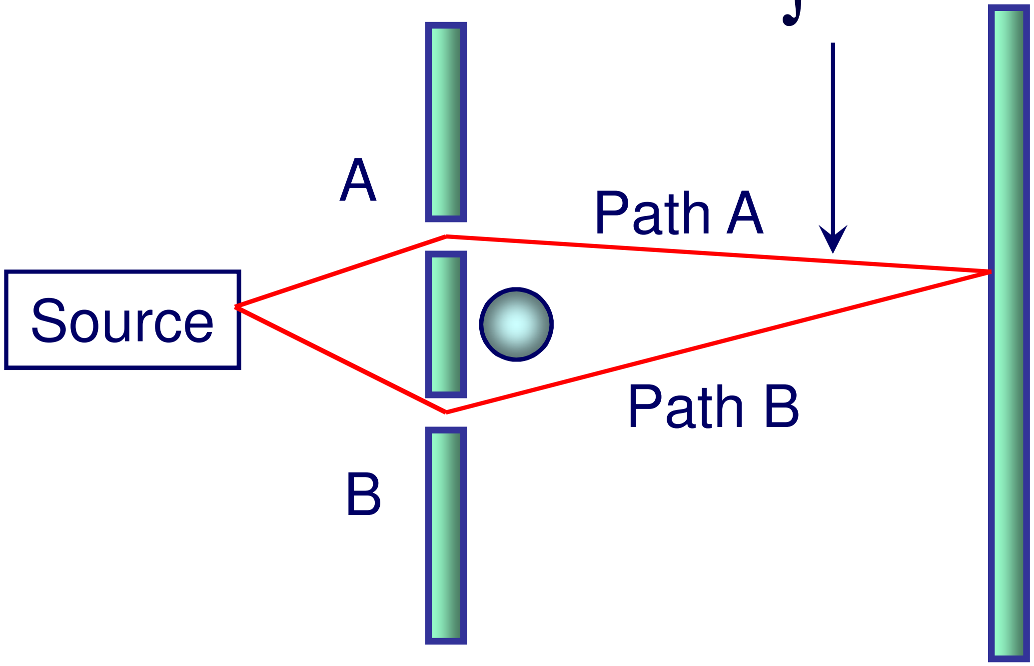
$$\psi_B = \psi_B^{(0)}\, e^{\frac{iq}{\hbar}\int_{path\ \mathrm{B}} \mathbf{A}(\mathbf{r})d\mathbf{r}}$$

The interference pattern on the screen is due to superposition of Ψ_A and Ψ_B.

Case 2: $\Phi \neq 0$

The interference experiment

$\oint \mathbf{dr}$ This integral encircles the solenoid



$$\int_{path\ \mathrm{A}} \mathbf{A}(\mathbf{r})\mathbf{dr} - \int_{path\ \mathrm{B}} \mathbf{A}(\mathbf{r})\mathbf{dr}$$

$$= \oint \mathbf{A}(\mathbf{r})\mathbf{dr} = \int_S [\nabla \times \mathbf{A}(\mathbf{r})] \mathrm{d}\mathbf{S}$$

$$= \int_S \mathbf{B}\, \mathrm{d}\mathbf{S} = \Phi$$

$$\psi = \psi_A + \psi_B = \psi_A^{(0)}\, e^{\frac{iq}{\hbar}\int_{path\ \mathrm{A}} \mathbf{A}(\mathbf{r})\mathbf{dr}} + \psi_B^{(0)}\, e^{\frac{iq}{\hbar}\int_{path\ \mathrm{B}} \mathbf{A}(\mathbf{r})\mathbf{dr}}$$

$$= \left[\psi_A^{(0)} e^{\frac{iq}{\hbar}\oint \mathbf{A}(\mathbf{r})\mathbf{dr}} + \psi_B^{(0)} \right] e^{\frac{iq}{\hbar}\int_{path\ \mathrm{B}} \mathbf{A}(\mathbf{r})\mathbf{dr}} = \left[\psi_A^{(0)} e^{\frac{iq}{\hbar}\Phi} + \psi_B^{(0)} \right] e^{\frac{iq}{\hbar}\int_{path\ \mathrm{B}} \mathbf{A}(\mathbf{r})\mathbf{dr}}$$

The Aharonov-Bohm effect

$$\psi = \psi_A + \psi_B = \left[\psi_A^{(0)} e^{\frac{iq}{\hbar}\Phi} + \psi_B^{(0)} \right] e^{\frac{iq}{\hbar}\int_{path\,B} \mathbf{A}(\mathbf{r})d\mathbf{r}}$$

The relative phase difference between Ψ_A and Ψ_B depends on the magnetic flux and the position of the interference pattern maxima are shifted due to the variations in magnetic flux even though the particles are never in the region where B≠0.

This is known as Aharonov-Bohm effect
Y. Aharonov, D. Bohm, Phys. Rev. 115, 485 (1959).

It has been experimentally verified by
R.G. Chambers in 1960.

$q=e$

Intensity of the electron density

$$I=\{\psi_A+\psi_B\}\{\psi_A+\psi_B\}^*$$

$$=\left\{\psi_A^{(0)}\, e^{\frac{ie}{\hbar}\int_{path\,\mathrm{A}}\mathbf{A}(\mathbf{r})d\mathbf{r}}+\psi_B^{(0)}\, e^{\frac{ie}{\hbar}\int_{path\,\mathrm{B}}\mathbf{A}(\mathbf{r})d\mathbf{r}}\right\}\left\{\psi_A^{(0)}\, e^{\frac{ie}{\hbar}\int_{path\,\mathrm{A}}\mathbf{A}(\mathbf{r})d\mathbf{r}}+\psi_B^{(0)}\, e^{\frac{ie}{\hbar}\int_{path\,\mathrm{B}}\mathbf{A}(\mathbf{r})d\mathbf{r}}\right\}^*$$

We assume $\psi_A^{(0)}=\psi_B^{(0)}=\psi^{(0)}$

$$I\propto\left\{e^{\frac{ie}{\hbar}\int_{path\,\mathrm{A}}\mathbf{A}(\mathbf{r})d\mathbf{r}-\frac{ie}{\hbar}\int_{path\,B}\mathbf{A}(\mathbf{r})d\mathbf{r}}+e^{-\frac{ie}{\hbar}\int_{path\,\mathrm{A}}\mathbf{A}(\mathbf{r})d\mathbf{r}+\frac{ie}{\hbar}\int_{path\,B}\mathbf{A}(\mathbf{r})d\mathbf{r}}\right\}$$

$$=\left\{e^{\frac{ie}{\hbar}\oint\mathbf{A}(\mathbf{r})d\mathbf{r}}+e^{-\frac{ie}{\hbar}\oint\mathbf{A}(\mathbf{r})d\mathbf{r}}\right\}\propto\cos\left[\frac{e}{\hbar}\oint\mathbf{A}(\mathbf{r})d\mathbf{r}\right]=\cos\left[\frac{e}{\hbar}\Phi\right]$$

Classical vs. quantum mechanics

Why is this effect so surprising?

Because particle never enters the region of non-zero magnetic field!

↓

The classical Lorentz force on the particle is zero and the classical trajectory would not be deflected.

In quantum mechanics, this remains true *on the average*.
The magnetic flux affects the motion of the individual particles but it produces zero average deflection.
The positions of the fringes shifts systematically as the flux is varied but the centroid of the diffraction pattern does not move.

Is magnetic flux quantized?

The relative phase of the Ψ_A and Ψ_B is given by $\exp\left(iq\Phi/\hbar\right)$.

If the magnetic flux were quantized in multiples of $2\pi\hbar/q$, then the phase factor would be 1and the interference pattern would not depend on the flux.

However, this happens not to be the case (as verified by the experiment) and magnetic flux is not quantized.

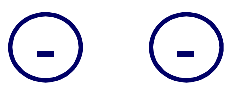
The exception is flux quantization in superconductors. It is, however, a property of the superconducting state and not a general property of the electromagnetic field.

Superconductivity

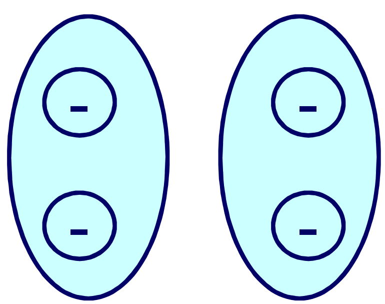
Superconductivity- the persistence of the resistantless electric currents.

Certain metals lose their resistance when the temperature is lowered below a certain critical temperature (which is different for different metals).

Main point of the theory, known as Bardeen-Cooper-Schrieffer (BCS) theory is that in normal metals the electrons behave as **fermions**, while in superconducting state they form “Cooper pairs” and behave like **bosons**.



Singe electrons- the wave function is antisymmetric under exchange



Cooper pairs - the wave function is symmetric under exchange

Superconductivity

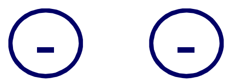
Normally electrons do not form pairs as they repel each other. However, inside the material the electrons interact with ions of the crystal lattice. Very simplify, the electron can interact with the positively charged background ions and create a local potential disturbance which can attract another electron.

The binding energy of the two electrons is very small, 1meV, and the pairs dissociate at higher temperatures.

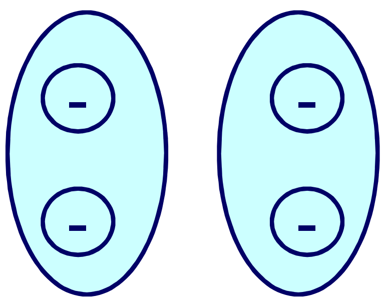
At low temperatures, the electrons can exists in the bound states (from Cooper pairs).

From BCS theory we learn that the lowest state of the system is the one in which Cooper pairs are formed.

Superconductivity



Singe electrons- only one electron can occupy a particular state

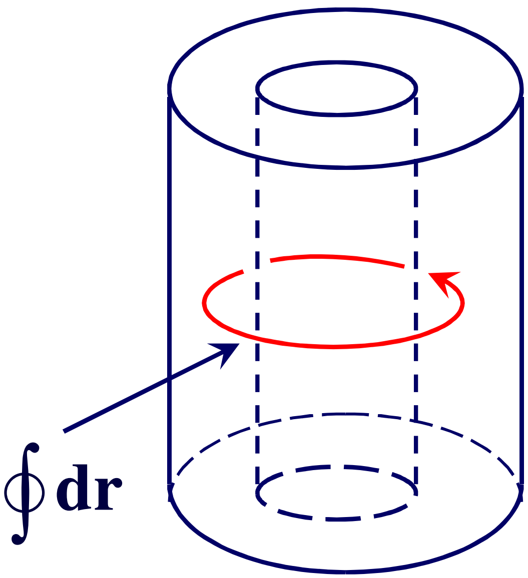


Cooper pairs – the above restriction no longer applies as electron pairs are bosons and very large number of pairs can occupy the same state

1. Therefore, the electron pairs do not have to move from an occupied state to unoccupied one to carry current.

2. The normal state is an excited state which is separated from the ground state (in which electrons form Cooper pairs) by an energy gap. Therefore, electrons do not suffer scattering which a source of resistance as there is an energy gap between their energy and the energies of the states to which they can scatter.

Flux quantization in superconductors



We consider a superconductor in form of a hollow cylinder which is placed in an external magnetic field, which is parallel to the axis of the cylinder.

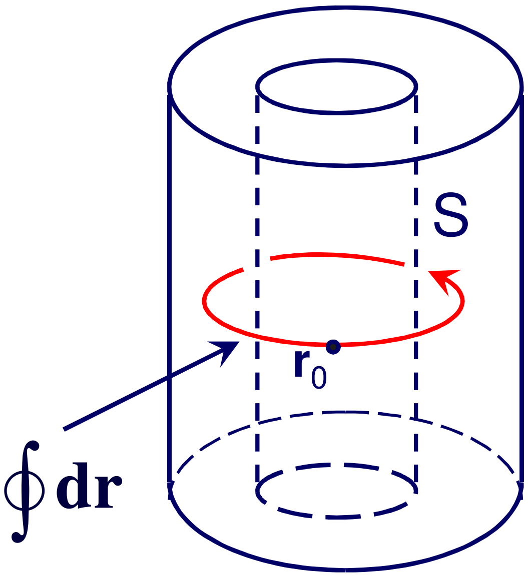
The magnetic field is expelled from the superconductor (Meissner effect) and vanishes within it. Therefore, Cooper pairs move in the region of **B**=0, and we can apply the results which we previously developed.

If the wave function of the Cooper pair in the absence of the field is $\psi^{(0)}$, then in the presence of the field we have

$$\psi'(\mathbf{r}) = \psi^{(0)}(\mathbf{r})\, e^{(i2e/\hbar)\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\,\mathbf{dr}'}$$

Flux quantization in superconductors

$$\psi'(\mathbf{r}) = \psi^{(0)}(\mathbf{r}) e^{(i2e/\hbar)\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\, d\mathbf{r}'}$$



When we consider a closed path S around the cylinder which starts at point $\mathbf{r}_0$ we get

$$\psi'(\mathbf{r}) = \psi^{(0)}(\mathbf{r}) e^{(i2e/\hbar)\oint \mathbf{A}(\mathbf{r}')\mathbf{dr}'} = \psi^{(0)}(\mathbf{r}) e^{i2e\Phi/\hbar}$$

As the electron wave function should not be multivalued as we go around the cylinder we get the condition

$$\frac{2e\Phi}{\hbar} = 2n\pi \rightarrow \boxed{\Phi = \frac{n\pi\hbar}{e}}, \qquad n = 0, 1, 2, \ldots$$

And the flux enclosed by the superconducting cylinder (or ring) is quantized!

This effect has been experimentally verified which confirmed that the current in superconductors is carried by the pair of the electrons and not the individual electrons.

How this effect can be used?

The main attraction of the Aharonov-Bohm effect is the possibility to use it in switching devices, i.e. to use the change in magnetic filed to change the state of the device from 0 to 1.

How much do we have to change the magnetic field to switch from the constructive to destructive electron interference?

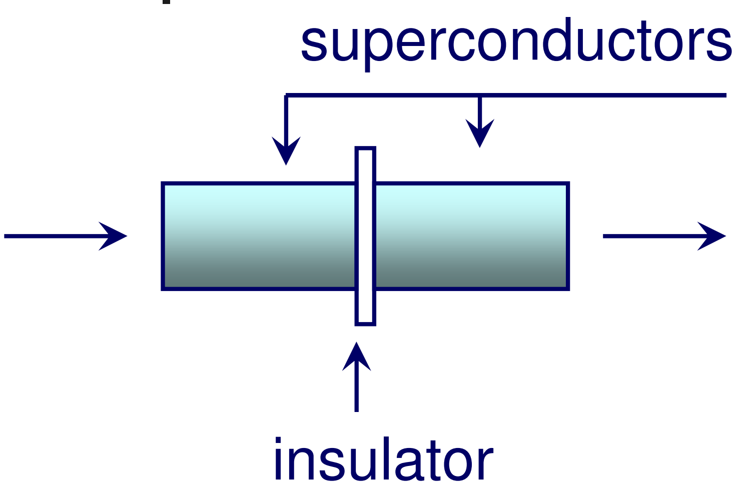
$$\Delta\Phi = \frac{\pi\hbar}{e}$$

$$\Delta B = \frac{\pi\hbar}{eA} \approx \frac{\pi \times 1.05 \times 10^{-34}\,\mathrm{J \cdot s}}{\left(1.6 \times 10^{-19} C\right)\left(20 \times 10^{-6} m^2\right)} \approx 5.1 \times 10^{-6} T$$

for 20μm x 20μm device

This is a very small field! The Earth's magnetic field is about 40μT.
It is very difficult to practically use.

Josephson junction



$$J = J_0 \sin\delta$$

Josephson junction:
a thin insulator sandwiched between two superconductors

phase difference $\delta = \theta_2 - \theta_1$

Depends on the tunneling probability of the electron pairs

There is a current flow across the junction in the absence of an applied voltage!

Superconducting devices

Extremely interesting devices may be designed with a superconducting loop with two arms being formed by Josephson junctions.

The operation of such devices is based on the fact that the phase difference around the closed superconducting loop which encloses the magnetic flux Φ is an integral product of $2e\Phi/\hbar$.

The current will vary with Φ and has maxima at $\dfrac{e\Phi}{\hbar} = n\pi$.

The control of the current through the superconducting loop is the basis for many important devices. Such loops may be used in production of low power digital logic devices, detectors, signal processing devices, and extremely sensitive magnetic field measurement instruments .

SQUID magnetometer (Superconductind QUantum Interference Device)