Lecture 8

Quantum teleportation

What is it?

Technique for moving quantum states around, even in an absence of quantum communication channel.

The problem:

Alice must deliver qubit $|\psi\rangle$ to Bob

- She does not know the state of the qubit
- She can use only classical channels

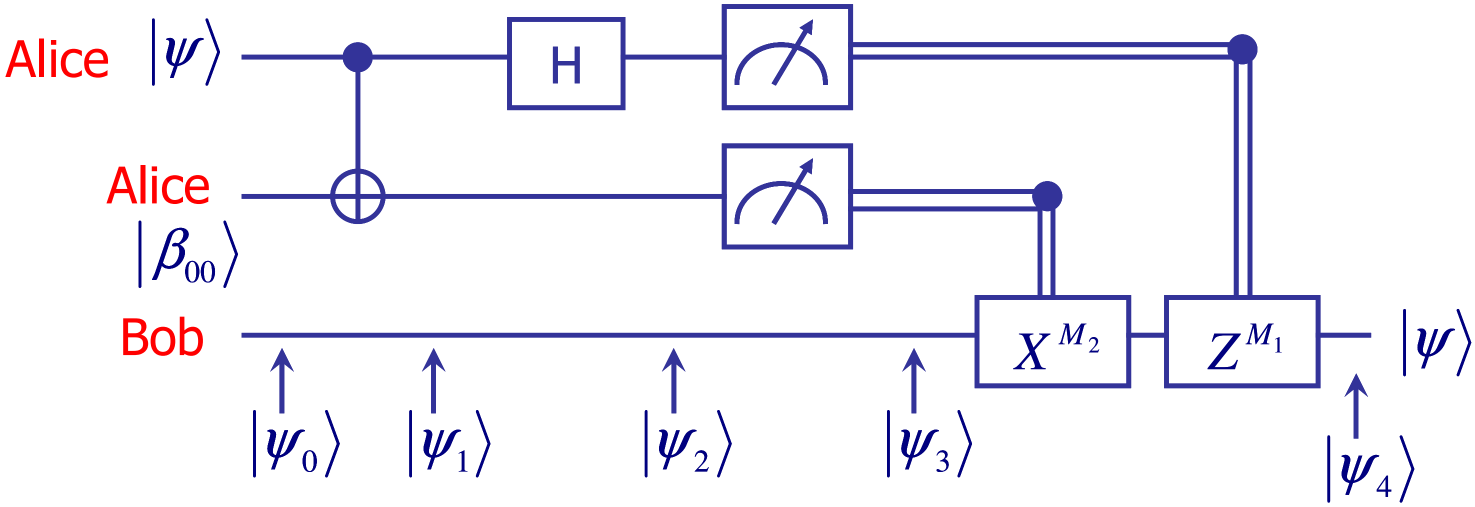
How does it work?

- Alice and Bob generate an EPR pair together.

EPR pair: two entangled qubits in the state $|\beta_{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

- The moved to different places and each took one qubit of the EPR pair.
- Alice interacts qubit $|\psi\rangle$ to be teleported with half of her EPR pair and then makes a measurement on two qubits which she has.
- She can get one out of four possible results: **00, 01, 10, and 11.**
- Alice reports this information to Bob.
- Bob performs one of four operations on his half of the EPR pair.
- Amazingly, he can recover the original state $|\psi\rangle$!

Teleportation scheme

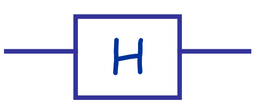


$$\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$$

Quantum circuit

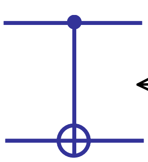
1. Each line represents one qubit.
Therefore, we have three qubits in our teleportation circuit.

2. If the gate is on a single line, this is a one-qubit gate applied to this one gubit only.



Hadamard gate is a single qubit gate that act only on qubit one in our circuit. The other two qubits are not affected by it.

3. If two lines are connected, it signifies the two qubit gate that acts of these two qubits only.



Quantum CNOT gate

4. Measurement is designated by sign 

We will now work out in detail how this circuit works.

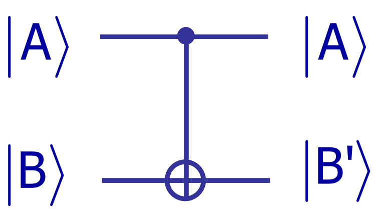
The initial state is

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle\left(|00\rangle+|11\rangle\right)+\beta|1\rangle\left(|00\rangle+|11\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\alpha|000\rangle+\alpha|011\rangle+\beta|100\rangle+\beta|111\rangle\right]$$

Next, we find the state $|\psi_1\rangle$.

The CNOT gate (control-NOT) works in the following way:



$\|AB\rangle$	$\|AB'\rangle$
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Logical operation: if A is in state |1>, flip the qubit B.

Class exercise: what is $|\psi_1\rangle$?

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[\alpha|000\rangle+\alpha|011\rangle+\beta|100\rangle+\beta|111\rangle\right]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle\right]$$

The next step involves Hadamard gate . First, we need to find out what Hadamard gate does to a qubit. The matrix for the Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Class exercise: If you operate by this gate on some unknown qubit, what will you get?

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Hint: you need to work out $H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \ldots$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\alpha\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|0\rangle} + \beta\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|1\rangle} + \alpha\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle} - \beta\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|0\rangle}\right)$$

$$= \alpha\,\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Another way to do it is to find how H acts on $|0\rangle$ and $|1\rangle$:

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Result:

$$\alpha|0\rangle+\beta|1\rangle \;—\boxed{H}—\; \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}}+\beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Therefore, the Hadamard gate corresponds to the following logical operation:

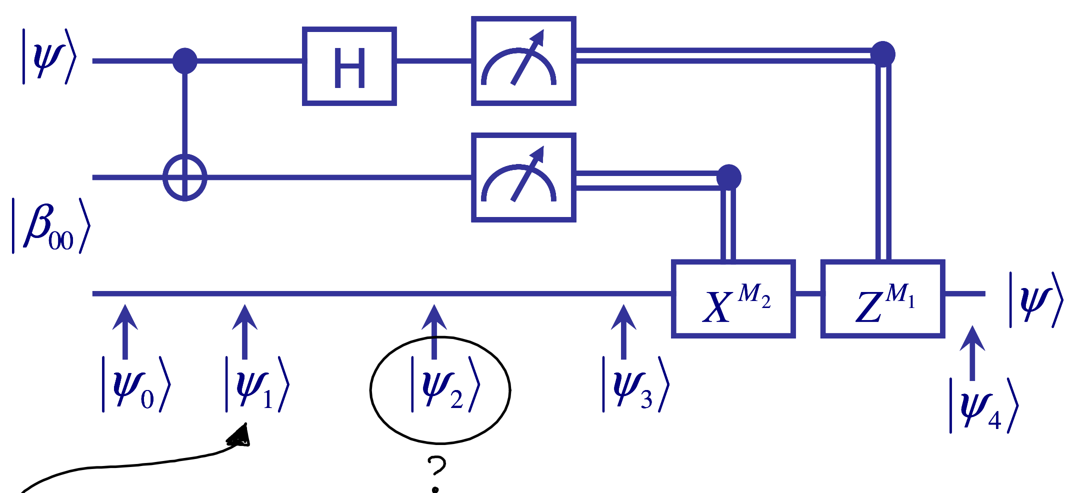
If the qubit is in state |0> it will become $\dfrac{|0\rangle+|1\rangle}{\sqrt{2}}$.

If the qubit is in state |1> it will become $\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}$.

Now we can work out the next step of our circuit.

Class exercise: what is $|\psi_2\rangle$? In your result, separate out the last third qubit and group together terms with the same states for the first two qubits , i.e. you result should be written as

$$|\psi_2\rangle = \frac{1}{2}\{\,|00\rangle(\qquad)+|01\rangle(\qquad)+|10\rangle(\qquad)+|11\rangle(\qquad)\,\}.$$



$$|\psi_1\rangle=\frac{1}{\sqrt{2}}\left[\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle\right]$$

$$H: \quad |0\rangle \longrightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |1\rangle \longrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left\{ \alpha H|0\rangle|00\rangle + \alpha H|0\rangle|11\rangle + \beta H|1\rangle|10\rangle + \beta H|1\rangle|01\rangle \right\}$$

$$= \frac{1}{2}\left\{ \alpha(|0\rangle+|1\rangle)|00\rangle + \alpha(|0\rangle+|1\rangle)|11\rangle + \beta(|0\rangle-|1\rangle)|10\rangle + \beta(|0\rangle-|1\rangle)|01\rangle \right\}$$

$$= \frac{1}{2}\left\{ \alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle \right\}$$

$$= \frac{1}{2}\left\{ \alpha|00\rangle|0\rangle + \alpha|10\rangle|0\rangle + \alpha|01\rangle|1\rangle + \alpha|11\rangle|1\rangle + \beta|01\rangle|0\rangle - \beta|11\rangle|0\rangle + \beta|00\rangle|1\rangle - \beta|10\rangle|1\rangle \right\}$$

$$= \frac{1}{2}\left\{ |00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle) \right\}$$

$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)$$
$$+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

Next step: measure the state of first two qubit. Possible results are |00>, |01>, |10>, and |11>. After Alice makes the measurement, she communicates the result to Bob.
Bob does the following conditional operation with his qubit:

1) If result is |00>, he does nothing: he already has the qubit |ψ> !
2) If result is |01>, he applies X gate.
3) If result is |10>, he applies Z gate.
4) If result is |11>, he applies both X and Z gate.

$$\alpha|0\rangle+\beta|1\rangle \;—\boxed{X}—\; \alpha|1\rangle+\beta|0\rangle$$

$|01\rangle \Rightarrow X(\alpha|1\rangle + \beta|0\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$

$$\alpha|0\rangle+\beta|1\rangle \;—\boxed{Z}—\; \alpha|0\rangle-\beta|1\rangle$$

$|10\rangle \Rightarrow Z(\alpha|0\rangle - \beta|1\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$

$|11\rangle \Rightarrow XZ(\alpha|1\rangle - \beta|0\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$