Lecture 7 **Quantum gates and Quantum circuits**

The standard quantum mechanical notation for a vector in a vector space is $|\psi\rangle$. ψ is a label for the vector.

$|.\rangle$ may be used to indicate that object is a vector (also called ket).

Vector space also contains zero vector.

We will denote it 0 instead of $|0\rangle$ since $|0\rangle$ is usually

used to designate $|\Psi_0\rangle \equiv |0\rangle$.

$$\boxed{\text{Qubit:} \quad \psi = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}}$$

A spanning set for a vector space: a set of vectors $|v_1\rangle,\ldots,|v_n\rangle$ such that any vector $|v\rangle$ in the vector space can be written as a linear combination $|v\rangle = \sum_i a_i |v_i\rangle$ of vectors in that set.

Example: a spanning set for the vector space C^2 is $|v_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

since any vector $|v\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ can be written as linear combination

$|v\rangle = \alpha|v_1\rangle + \beta|v_2\rangle$.

Note: a vector space may have many different spanning sets.

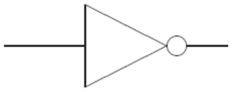
Example: $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

Any linear combination $\sum_i \alpha_i |\psi_i\rangle$ is a **superposition**

of the states $|\psi_i\rangle$ with amplitude α_i for the state $|\psi_i\rangle$.

Example: state $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ is a superposition of |0> and |1>.

Classical single bit gate: NOT gate



A	NOT A
0	1
1	0

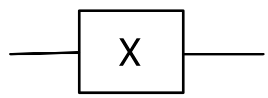
This is the only non-trivial single bit classical gate.

Quantum single qubit gates

Quantum NOT gate $\quad \alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|1\rangle + \beta|0\rangle$

Matrix representation for Quantum NOT (X gate)

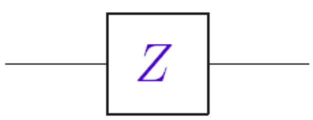
$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$



X gate

Z gate: leaves |0> unchanged and flips the sign of |1>

$\alpha|0\rangle + \beta|1\rangle$



$\alpha|0\rangle - \beta|1\rangle \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$Z\psi = Z\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \alpha|0\rangle - \beta|1\rangle$$

Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{H}—\; \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Exercise: use matrix form of H to demonstrate how Hadamard gate operate on a qubit

$$H\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} = \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

More single qubit gates

Note: matrix U describing singe qubit gate must be unitary. $U^+ U = 1.$

Phase gate: $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

Question to the class: what operation does this gate perform?

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{S}—\; ?$$

$$S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(leaves } |0\rangle \text{ unchanged)}$$

$$S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} \quad |1\rangle \to i|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{S}—\; \alpha|0\rangle + i\beta|1\rangle$$

$$\frac{\pi}{8} \text{ gate: } \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{pmatrix}$$

Question to the class:

Does this gate preserve normalization of a qubit? $\langle \psi | \psi \rangle = 1$

$$(\alpha^* \ \beta^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2 = 1$$

$$T \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta e^{i\pi/4} \end{pmatrix} \Rightarrow \quad |\alpha|^2 + |\beta|^2 e^{-i\pi/4} e^{i\pi/4} = 1$$

Note: all gates do since $U^+ U = 1$.

Why the T gate is called $\pi/8$ while $\pi/4$ appears in the definition?

The reason is historical. This gate is equivalent (up to unimportant global factor) to the following gate:

$$T = \exp\left(\frac{i\pi}{8}\right) \begin{pmatrix} \exp\left(-\frac{i\pi}{8}\right) & 0 \\ 0 & \exp\left(\frac{i\pi}{8}\right) \end{pmatrix}.$$

<u>Geometrical representation of a qubit</u>

We can re-write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ as

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi} \sin\frac{\theta}{2}|1\rangle \right)$$

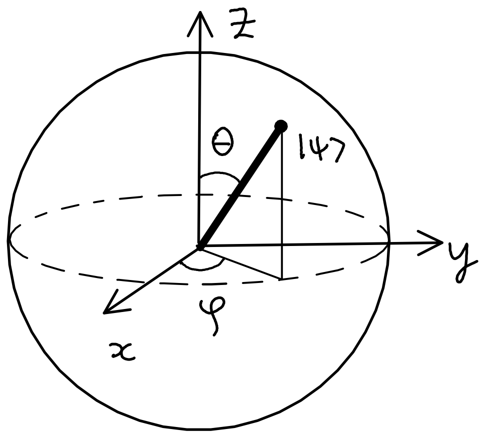
where $\gamma, \theta, \text{and } \varphi$ are real numbers.

$$|\alpha|^2 + |\beta|^2 = e^{-i\gamma} e^{i\gamma} \cos^2\frac{\theta}{2} + e^{i\gamma} e^{-i\gamma} e^{i\varphi} e^{-i\varphi} \sin^2\frac{\theta}{2} = 1$$

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

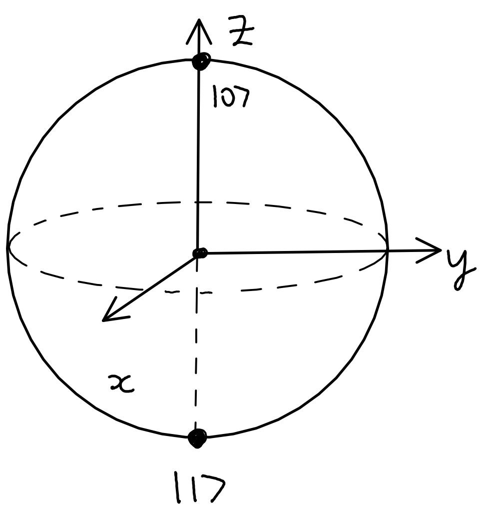
global factor can be omitted since it has no observable effects

The numbers θ and φ define a point on the unit three-dimensional sphere:



This sphere is called the Bloch sphere; it provides means of visualizing the state of a single qubit. The single qubit gates may be represented as rotations of a qubit on the Bloch sphere. Unfortunately, there is no simple generalization of the Bloch sphere for multiple qubits.

Class exercise: show $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$ on the Bloch sphere.



1) $|\psi\rangle = |0\rangle$

$\alpha = 1 \qquad \cos\frac{\theta}{2} = 1 \quad \theta = 0$

2) $|\psi\rangle = |1\rangle$

$\cos\frac{\theta}{2} = 0 \qquad \theta = \pi$

$\sin\frac{\pi}{2} = 1$

Rotation operators

The Pauli matrices X, Y, and Z give rise to the rotation operators about the x, y, and z axis of the Bloch sphere:

$$
\begin{aligned}
R_x(\theta) &\equiv e^{-i\theta X/2} = \begin{bmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \\
R_y(\theta) &\equiv e^{-i\theta Y/2} = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \\
R_z(\theta) &\equiv e^{-i\theta Z/2} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}
\end{aligned}
$$

Multiple qubits

The tensor product: way of putting vector spaces together to form larger vector spaces. (Example: to construct multiparticle systems).

V & W: vector spaces of dimension m and n, respectively, let us also assume that V and W are Hilbert spaces. Then, $V\otimes W$ is an mn dimensional vector space. The elements of $V\otimes W$ are linear combinations of $|v\rangle\otimes|w\rangle$.

Notations we use in this course:

$$|v\rangle\otimes|w\rangle\equiv|v\rangle|w\rangle\equiv|v,w\rangle\equiv|v\,w\rangle$$

Remember: $|\Psi\rangle=\frac{|00\rangle+|11\rangle}{\sqrt{2}}$).

Example: if V is a two-dimensional vector space with basis vectors

$|0\rangle$ and $|1\rangle$, then $|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\equiv|00\rangle+|11\rangle$

is an element of $V\otimes V$.

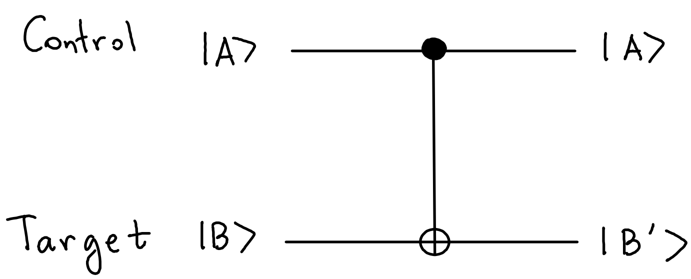
Useful notation: $|\psi\rangle^{\otimes k}=\underbrace{|\psi\rangle\otimes|\psi\rangle\ldots\otimes|\psi\rangle}_{k}$.

For example, $|\psi\rangle^{\otimes 2}=|\psi\rangle\otimes|\psi\rangle$.

TWO-QUBIT GATES

Controlled operations: "If A is true, then do B"

Controlled-NOT (CNOT) gate



$\|AB\rangle$	$\|AB'\rangle$
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Gate operations: if control qubit is $|1\rangle$, then flip the target qubit.

Question for the class: what is the matrix representation for this gate?

$$U|00\rangle = |00\rangle$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \\ \alpha_{41} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\alpha_{11} = 1$

$\alpha_{21} = \alpha_{31} = \alpha_{41} = 0$

In the same way $\alpha_{22}=1,\ \alpha_{12}=\alpha_{32}=\alpha_{42}=0$ from $U|01\rangle=|01\rangle$

$U|10\rangle=|11\rangle \Rightarrow$

$$\begin{pmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \\ \alpha_{43} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$$

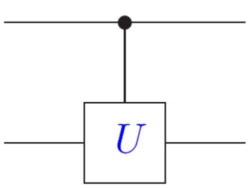
$$\alpha_{33}=\alpha_{13}=\alpha_{23}=0, \quad \boxed{\alpha_{43}=1}$$

$U|11\rangle=|10\rangle \Rightarrow \quad \alpha_{14}=\alpha_{24}=\alpha_{44}=0 \quad \alpha_{34}=1$

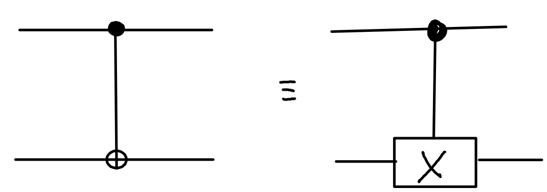
$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

More on controlled operations

Suppose U is an arbitrary single qubit unitary operation. A controlled-U operation is a two-qubit operation with a control qubit and a target qubit. If control qubit is set, then U is applied to the target qubit.

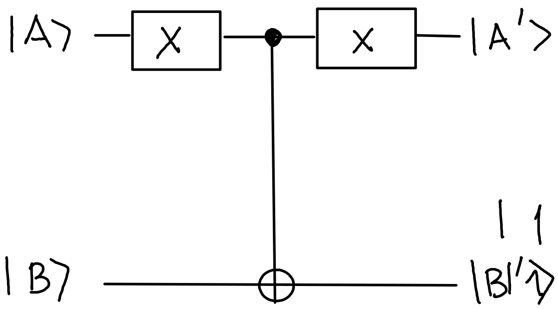


Example: controlled-NOT gate is controlled-X gate.



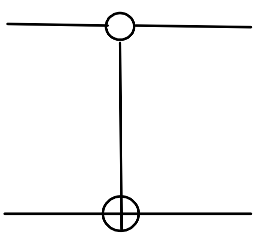
Note that now we can write its matrix right away: $U = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$

Example: what does this circuit do?

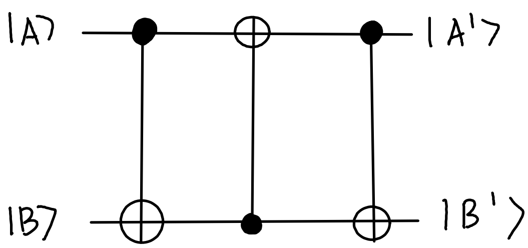


	X		CNOT		X	
$\lvert 00\rangle$	$\to$	$\lvert 10\rangle$	$\to$	$1\rangle$	$\to$	$\lvert 0\boxed{1}\rangle$
$\lvert 01\rangle$	$\to$	$\lvert 11\rangle$	$\to$	$0\rangle$	$\to$	$\lvert 0\boxed{0}\rangle$
$\lvert 10\rangle$	$\to$	$\lvert 00\rangle$	$\to$	$\lvert 00\rangle$	$\to$	$\lvert 10\rangle$
$\lvert 11\rangle$	$\to$	$\lvert 01\rangle$	$\to$	$\lvert 01\rangle$	$\to$	$\lvert 11\rangle$

flip target qubit if control qubit is $|0\rangle$



Question for the class: what does this circuit do?

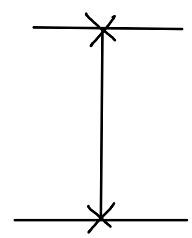


c t		t c		c t		
$\lvert 00\rangle$	$\to$	$\lvert 00\rangle$	$\to$	$\lvert 00\rangle$	$\to$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\to$	$\lvert 01\rangle$	$\to$	$\lvert 11\rangle$	$\to$	$\lvert 10\rangle$
$\lvert 10\rangle$	$\to$	$\lvert 11\rangle$	$\to$	$\lvert 01\rangle$	$\to$	$\lvert 01\rangle$
$\lvert 11\rangle$	$\to$	$\lvert 10\rangle$	$\to$	$\lvert 10\rangle$	$\to$	$\lvert 11\rangle$

$|A\rangle \to |B\rangle$

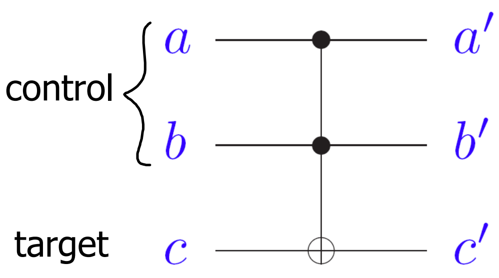
$|B\rangle \to |A\rangle$

"swap circuit"



Classical computation on a quantum computer

Toffoli gate



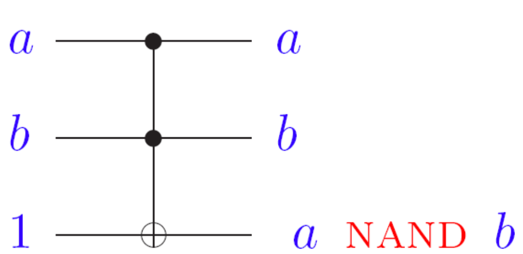
a	b	c	a'	b'	c'
0	0	1	0	0	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	1	0
0	1	0	0	1	0
0	0	0	0	0	0
1	0	0	1	0	0
1	1	0	1	1	1

If both a & b are $|1\rangle$, flip the target qubit.

Questions for the class:

1) How would you use Toffoli gate to implement NAND gate?
2) How would you use Toffoli gate to make a "copy"?

1)

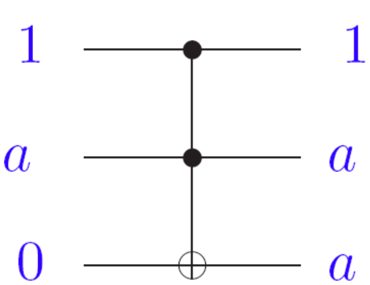


a NAND b

a	b	c	a'	b'	c'
0	0	1	0	0	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	1	0
0	1	0	0	1	0
0	0	0	0	0	0
1	0	0	1	0	0
1	1	0	1	1	1

use this part

2)



a	b	c	a'	b'	c'
0	0	1	0	0	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	1	0
0	1	0	0	1	0
0	0	0	0	0	0
1	0	0	1	0	0
1	1	0	1	1	1
	a			a	a

use this part

Universal quantum gates

A set of gates is said to be **universal for quantum computation** if any unitary operation may be approximated to arbitrary accuracy by a quantum circuit involving only those gates.

A unitary matrix U which acts on d-dimensional Hilbert space may be decomposed into a product of two-level matrices; i.e. unitary matrices which act non-trivially only on two-or-fewer vector components.

$$U = V_1 \dots V_k$$

$$k \leq d(d-1)/2$$