Lecture 15

Quantum parallelism

f(x): {0,1} → {0,1} is a function with one-bit domain and range.

Example: #1 f(x) = 0, 0 → 0, 1 → 0

Question for the class: list all possible f(x) functions.

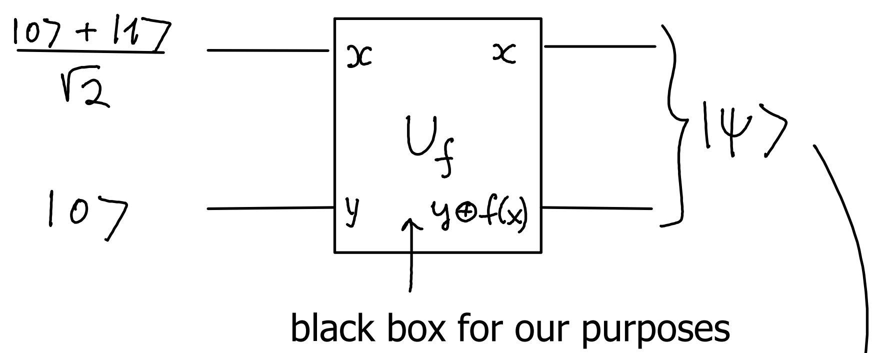
(#2) f (x) = 1, 0 → 1, 1 → 1
(#3) bit flip 0 → 1, 1 → 0
(#4) f(x)=x, 0 → 0, 1 → 1

Circuit for evaluating both f(0) and f(1) in one step

Note that x⊕y is addition modulo 2, i.e. 0⊕0=0, 0⊕1=1, 1⊕0=1, 1⊕1=0

(a) data register

(b) target register

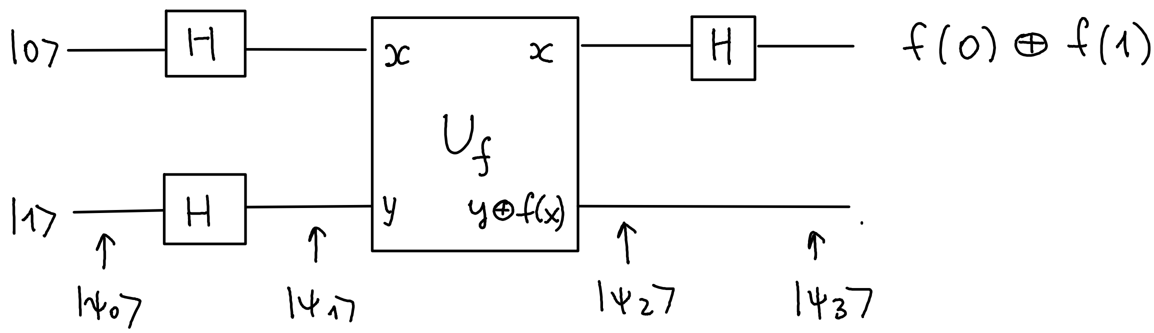


Since $0 \oplus f(x) = f(x)$ the output is

$$|\psi\rangle = |x, 0 \oplus f(x)\rangle = |x, f(x)\rangle = \frac{|0, f(0)\rangle + |1, f(1)\rangle}{\sqrt{2}}$$

"Quantum parallelism": information about both f(0) and f(1) is obtained in a single evaluation of the function f (x).

Deutch's algorithm: evaluating f(0)⊕f(1)
(pages 30-34 of the textbook)



This circuit evaluates f(0)⊕f(1) that is global property of f(x) with only one evaluation of f(x) function.

$$|\psi_0\rangle = |01\rangle$$

Class exercise: write $|\psi_1\rangle$.

$$H: \quad |0\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |1\rangle \to \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

To write out $|\psi_2\rangle$ we need to know how U_f box will act on our wave function $|\psi_1\rangle$.
Below, we prove that

$$\boxed{U_f\left[|x\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}\right] \to (-1)^{f(x)} \, |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}}$$

<u>Proof:</u>

$$U_f: \quad |x, y\rangle \to |x, y \oplus f(x)\rangle$$

$$y \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$U_f: \quad |x, 0\rangle \rightarrow |x, 0 \oplus f(x)\rangle = |x, f(x)\rangle$$

$$U_f\ |x,1\rangle \rightarrow |x, 1\oplus f(x)\rangle$$

Therefore, $U_f : |x, (|0\rangle - |1\rangle)\rangle = |x, f(x)\rangle - |x, 1\oplus f(x)\rangle$

We will evaluate $|x, f(x)\rangle - |x, 1\oplus f(x)\rangle$

for one of the functions f(x) as an example.

$\#1:\ f(x) = 0,\ 0\rightarrow 0,\ 1\rightarrow 0$

$$\left.\begin{array}{ll} x=|0\rangle & |x, f(x)\rangle - |x, 1\oplus f(x)\rangle = |0,0\rangle - |0,1\rangle \\ x=|1\rangle & |x, f(x)\rangle - |x, 1\oplus f(x)\rangle = |1,0\rangle - |1,1\rangle \end{array}\right\} = (*)$$

$$(*) = \left.\begin{array}{l} \overset{\downarrow x}{|0\rangle}(|0\rangle - |1\rangle) \\ \underset{\uparrow x}{|1\rangle}(|0\rangle - |1\rangle) \end{array}\right\} = |x\rangle(|0\rangle - |1\rangle) = \underbrace{(-1)^{f(x)}}_{=1 \text{ since } f(x)=0} |x\rangle(|0\rangle - |1\rangle)$$

Q.E.D

<u>Class exercise:</u> prove that for other possible f(x) functions (see page 1), we get

$$|x, f(x)\rangle - |x, 1\oplus f(x)\rangle = (-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle)$$

If $f(x) = 1$, then $1\oplus f(x) = 0 \Rightarrow$

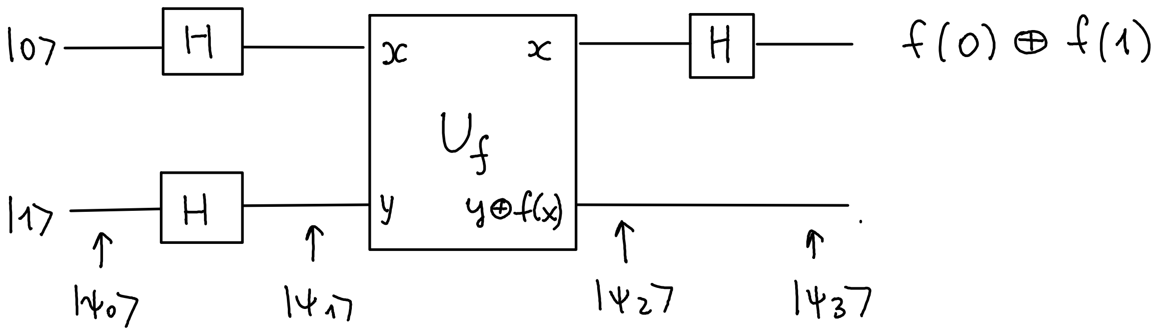
$$|x, f(x)\rangle - |x, 1\oplus f(x)\rangle = |x,1\rangle - |x,0\rangle = (-1)|x\rangle(|0\rangle - |1\rangle)$$

If $f(x) = 0$, then $1\oplus f(x) = 1 \Rightarrow$

$$|x, f(x)\rangle - |x, 1\oplus f(x)\rangle = |x,0\rangle - |x,1\rangle = |x\rangle(|0\rangle - |1\rangle)$$

Therefore, $|x, f(x)\rangle - |x, 1\oplus f(x)\rangle = (-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle)$

You can also check it by plugging in actual values of x and f(x) for remaining three cases.



Therefore, we proved that

$$U_f \left[|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \to (-1)^{f(x)} \; |x\rangle \; \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|x\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

plug here

$$|\psi_2\rangle = \left(\frac{(-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$|\psi_2\rangle = \begin{cases} \pm \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] & \text{if } f(0) = f(1) \\ & \text{(same signs of } (-1)^{f(0)} \text{ and } (-1)^{f(1)}\text{)} \\ \pm \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] & \text{if } f(0) \neq f(1) \\ & \text{(opposite signs of } (-1)^{f(0)} \text{ and } (-1)^{f(1)}\text{)} \end{cases}$$

Applying Hadamard gate to the first qubit gives

$$|\psi_3\rangle = \begin{cases} \pm|0\rangle\left[\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) = f(1) \\ \pm|1\rangle\left[\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}\right] & \text{if} \quad f(0) \neq f(1) \end{cases}$$

$$f(0) \oplus f(1) = 0 \quad \text{if} \quad f(0) = f(1) \quad [0,0 \text{ or } 1,1]$$

$$f(0) \oplus f(1) = 1 \quad \text{if} \quad f(0) \neq f(1) \quad [0,1 \text{ or } 1,0]$$

Therefore,

$$|\psi_3\rangle = \pm\,|f(0) \oplus f(1)\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Therefore, measurement on the first qubit determines f(0) ⊕ f(1), a global property of f(x) using only one evaluation of f(x)!