Lecture 11

Quantum error correction

More on syndrome measurement

Suppose that instead of measuring the four projectors P_0, P_1, P_2, and P_3, we performed two measurements, the first of the observable Z_1Z_2 (i.e. $Z \otimes Z \otimes I$)

measuremt on the first qubit

measurement on the second qubit

nothing is done to the third qubit

and the second of the observable Z_2Z_3 ($I \otimes Z \otimes Z$).

measurement on the second qubit

measurement on the third qubit

Each of these observables has eigenvalues ± 1, so each measurement provides a single bit of information, total two bits of information $\rightarrow$ four possible syndromes.

The first measurement can be thought of as comparing the first and second qubit to se if they are the same and the second measurements is equivalent to comparing the second and third qubit.

Reminder: $Z = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}$ $\quad \alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle$

Spectral decomposition is $\quad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Therefore, the spectral decomposition of Z_1Z_2 operator is

$$Z_1Z_2 = (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes I$$

$$= (\underbrace{|0\rangle \otimes |0\rangle \;\; \langle 0| \otimes \langle 0|}_{|00\rangle\langle 00|} - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|) \otimes I$$

$$= (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I \qquad \text{(same)}$$

$$- (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I \qquad \text{(different)}$$

which corresponds to projective measurements with projectors

$$① \quad (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I$$
$$② \quad (|01\rangle\langle 10| + |10\rangle\langle 10|) \otimes I$$

Therefore, of both qubits (1 and 2) are the same, we get +1, if they are different we get -1.

Measuring Z_2Z_3 compares the values of the second and third qubits. Again, we get +1 if they are the same and -1 if they are different.

Combining these results, we can find out what happened (assuming that error only happened on one qubit.

Z_1Z_2	Z_2Z_3		
+1	+1	no bit flip	with high probability as we exclude 2 bit flips
+1	-1	third qubit flipped	
-1	+1	first qubit flipped	
-1	-1	second qubit flipped	

The three qubit phase flip code (no classical analog)

Sending the qubit through a channel that flips the relative phase of the qubit with probability p. There is no classical equivalent to a phase flip channel.

$$|\psi\rangle \xrightarrow{p} Z|\psi\rangle$$

The state $|\psi\rangle$ is taken to the state $Z|\psi\rangle$ with probability p.

$$\alpha|0\rangle + \beta|1\rangle \; -\boxed{Z}- \; \alpha|0\rangle - \beta|1\rangle$$

Phase flip channel may be turned into a bit flip channel if we work in basis

$$|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

The operator Z acts as a bit flip with respect to this basis:

$$Z|+\rangle = Z|(|0\rangle + |1\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv |-\rangle$$

$$Z|-\rangle = Z|(|0\rangle - |1\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle$$

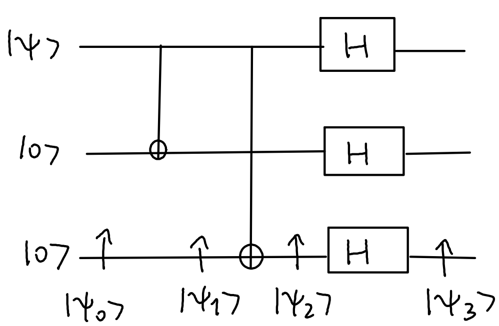
Therefore, we can use states

$$|0_L\rangle \equiv |+++\rangle$$

$$|1_L\rangle \equiv |---\rangle$$

to encode the information. All operations are performed with respect to |+> |-> basis.

The corresponding logical circuit for the encoding is



$$-\boxed{H}- \quad |0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \equiv |+\rangle$$

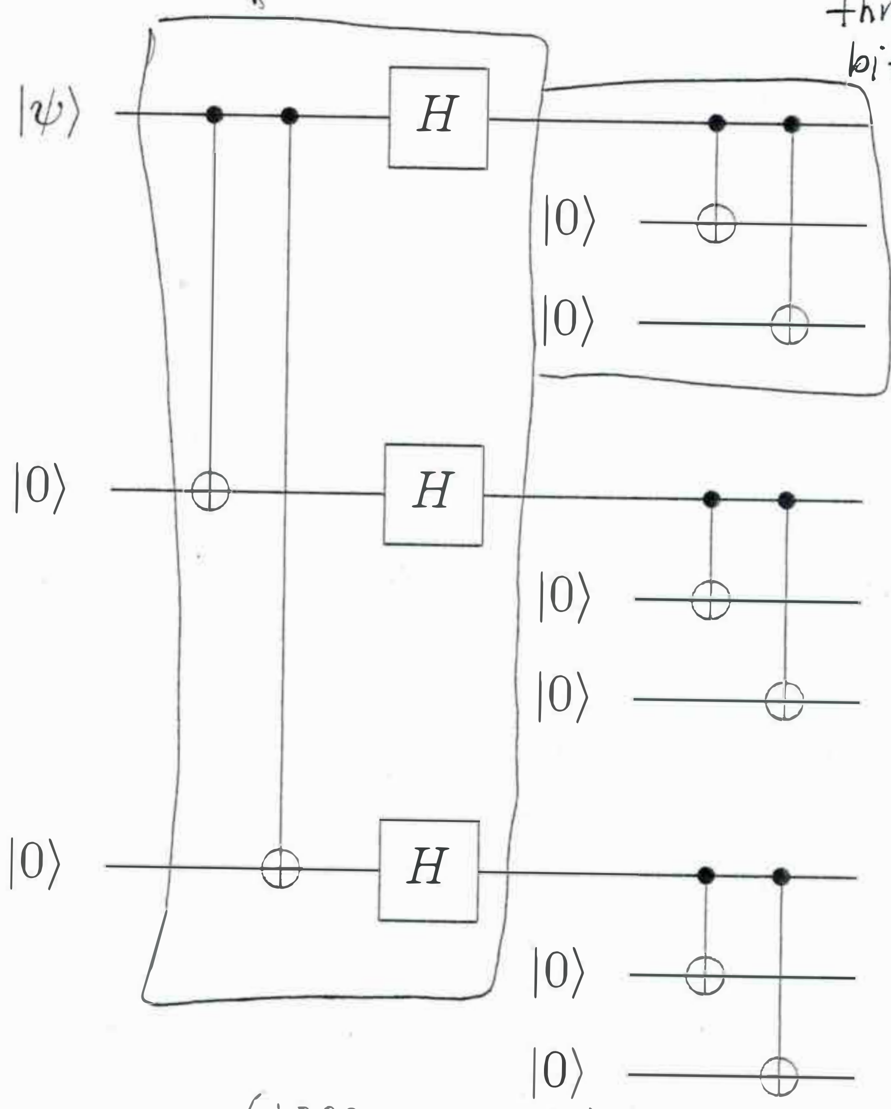
$$|1\rangle \rightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}} \equiv |-\rangle$$

Class exercise: verify that this circuit does the required encoding.

three qubit phase flip code

Shor's code

three qubit bit flip code



Such method (encoding using a hierarch of levels is know as **concatenation**

$$|0\rangle \to |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \to |0_L\rangle = \frac{(|000\rangle - |111\rangle)(|000 - 111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

$$\alpha|0\rangle + \beta|1\rangle \to \alpha|0\rangle_L + \beta|1_L\rangle$$

Suppose a bit flip occured on the first qubit. We perform measurement $Z_1 Z_2$ to compare first and ~~second~~ second qubit. We find that they are different. Making $Z_2 Z_3$ measurement will show that 2 & 3 are the same, so the error could not have been on the second qubit. Then, we recover the error by flipping the first qubit back.

The phase flips are handled in the same way. The phase flip on <u>any</u> of the first three qubits will flip the sign of the first block; i.e.

$$|000\rangle + |111\rangle \mapsto |000\rangle - |111\rangle$$

due to phase flip on q#1, 2, or 3.

Syndrome measurement is done by comparing signs of <u>blocks</u> of 3 qubits. Here, we find that blocks 1 & 2 have different signs but blocks 2 & 3 have the same signs $\Rightarrow$ the error (phase flip) occured on the one of the first three qubits.

Syndrome measurement for the ~~f~~ phase flips:

$X_1 X_2 X_3 X_4 X_5 X_6$ and $X_4 X_5 X_6 X_7 X_8 X_9$.

Let's verify it:

① Two blocks are the same (same phase)

$X_1 X_2 X_3 X_4 X_5 X_6 \equiv X_{1-6}$

$|\psi\rangle = \frac{1}{2}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \Rightarrow$

$X_{1-6}\psi = \frac{1}{2}(|111\rangle + |000\rangle)(|111\rangle + |000\rangle) \Rightarrow$

$\langle\psi| X_{1-6} |\psi\rangle = 1$

② If two blocks have ~~to~~ different phases

$|\Psi_1\rangle = \frac{1}{2}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)$, then

$X_{1-6}|\Psi_1\rangle = \frac{1}{2}(|111\rangle + |000\rangle)(|111\rangle - |000\rangle)$

$= -\frac{1}{2}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle) \Rightarrow$

$\langle\Psi_1| X_{1-6} |\Psi_1\rangle = -1$

Same result when we consider our complete wave function $\alpha|0\rangle_L + \beta|1\rangle_L$

Therefore, we will come the ~~th~~ same situation we hade with 3 qubit phase flip code:

① Compare 1 & 2 blocks using $X_1X_2X_3X_4X_5X_6$

② Compare 2 & 3 blocks using $X_4X_5X_6X_7X_8X_9$

The four possible results are

X_{1-6}	X_{4-9}	
+1	+1	no error
+1	−1	phase flip on block #3
−1	+1	phase flip on block #1
−1	−1	phase flip on block #2

with high probability

The recovery may be accomplished by operator $Z_1Z_2Z_3$ for phase flip on any of the first three qubits, etc.

Now, suppose both bit and phase flip occured to the first qubit (equivalent to operator Z_1X_1. Then, we simply ~~to~~ do the detection in sequence.

First, detect the bit flip and correct (1-3)

Second, detect the phase flip on block 1 and correct.

Shor's code protect from arbitrary errors of 1 qubit.

No additional work need to be done.

Suppose the noise of arbitrary type occures first qubit

Describe the noise by a trace-preserving quantum operator $\mathcal{E}$

We expand $\mathcal{E}$ in an operator-sum representation with ~~elements~~ operation elements $\{E_i\}$

State of $|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$ (before the noise)

After the noise acted the state is

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i |\psi\rangle\langle\psi| E_i^\dagger$$

E_i is operator on the first qubit alone and may be exanded as linear combination

$$E_i = e_{i0} I + e_{i1} X_1 + e_{i2} Z_1 + e_{i3} X_1 Z_1$$

There for, (un-normalized) state $E_i|\psi\rangle$ ~~...~~ is a superposition of four states,

$\boxed{|\psi\rangle,\ X_1|\psi\rangle,\ Z_1|\psi\rangle \text{ and } X_1 Z_1|\psi\rangle}$

Measuring the error syndrom will collapse this superposition into one of the four states; recovery may be performed to correct to get state $|\psi\rangle$ back.

the same is true for all E_i

Therefore, correcting the discrete set of errors will automatically correct much larger (continuous) class of errors.

<u>What about noise on multiple qubits?</u>

(1) We can generally assume that noise acts on qubits independently. If effet of noise on one qubit is ~~sma~~ small we can expand the total effect of noise as sum of terms:

$$\underbrace{\boxed{\text{~~noise~~ error on on qub.}} + \boxed{\text{error on 1 qubit}}}_{\text{Dominating terms}} + \boxed{\text{errors on 2 qubits}} \ldots$$

(2) If (1) does not apply then we can construct error-correcting codes which ~~cot~~ ~~on~~ can correct errors on more then a singe qubit.