Lecture 10

Quantum error correction

Classical error correction

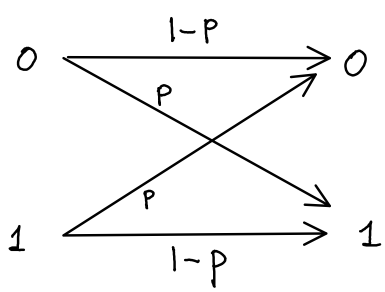
Modern computers: failure rate is below one error in 10^{17} operations
Data transmission and storage (file transfers, cell phones, DVD, CD, etc.): noise problems

Solution: use error-correcting codes to protect against the effect of noise.
Key idea: encode the message with redundant information. Redundancy in the encoded message allows to recover the information in the original message.

Classical error correction: repetition code

Example: sending one bit of information across noisy channel.
Effects of the noise: flip the bit with probability p.

Binary symmetric channel:



Solution: protect the information by making three copy of the bit:

$0 \rightarrow 000$
$1 \rightarrow 111$

Decoding: "majority voting" $\boxed{011}$ 0 1
vote 1 : 2 therefore, flip 0 back to 1: $\boxed{001 \rightarrow 111}$

Note: in this scheme it is impossible to recover the information correctly if more than one bit is flipped.

Probability of error in the non-encoded message: **p.**

Question for the class: what is the probability of error in the three-bit repetition code if the probability of the bit flip is p?

(1) Probability that two bits are flipped:

probability p

probability (1-p)
[correct transmission]

$$000 \rightarrow \begin{cases} 1\,1\,0 \\ 1\,0\,1 \\ 0\,1\,1 \end{cases}$$

Total probability is $3p^2(1-p)$.

(2) Probability that three bits are flipped is p^3: $000 \longrightarrow 111$ (p p p).

Total probability of error p_e is $p_e=3p^2(1-p) + p^3 = 3p^2 - 2p^3$

Question for the class: under what circumstances three-bit repetition code is less reliable than the original unencoded transmission?

$p_e > p$ if $3p^2 - 2p^3 > p \longrightarrow p > 1/2$

Therefore, three - bit repetition code is less reliable than original one-bit transmission if error rate for a single bit flip exceeds 1/2.

Many clever ideas have been developed for error-correcting codes, always involving redundancy.

Quantum error correction

Major difficulties:

(1) No cloning: we can not just duplicate qubit three or more times as in the classical case.

(2) Errors are continuous:
A continuum of different errors may occur on a single qubit. Therefore, determination of which error occurred requires infinite precision and infinite resources.

(3) Measurement collapses the wave function; recovery is impossible if quantum superposition is destroyed.

None of these problems is fatal and all issues may be overcome.

The three qubit bit flip code

Example: sending the qubit through a channel which flips qubit with probability p.
The state $|\psi\rangle$ is changed to state X $|\psi\rangle$.

Reminder: X gate switches $|0\rangle$ and $|1\rangle$

$$X: \quad \alpha|0\rangle + \beta|1\rangle \xrightarrow{X} \alpha|1\rangle + \beta|0\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$\boxed{|\psi\rangle \xrightarrow{p} X|\psi\rangle \qquad (\text{bit flip})}$$

How does three bit flip code work?

We encode the qubit $\alpha|0\rangle + \beta|1\rangle$ in three qubits as follows:

$$\boxed{\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|000\rangle + \beta|111\rangle}$$

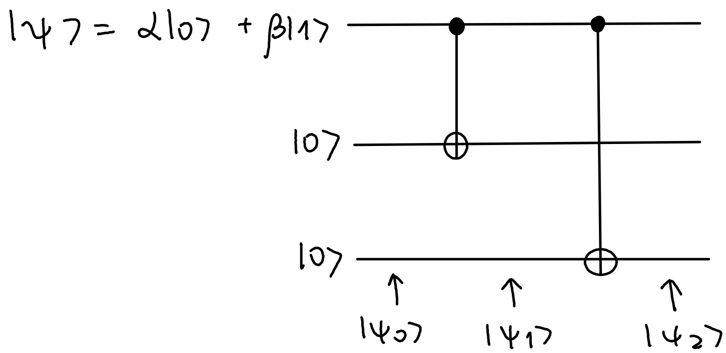
$$|0\rangle \rightarrow |0\rangle_L \equiv |000\rangle$$

$$|1\rangle \rightarrow |1\rangle_L \equiv |111\rangle$$

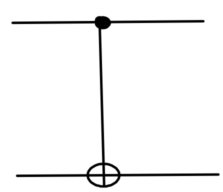
"Logical qubits"

Superpositions of basis states are taken to corresponding superpositions of encoded states.

Exercise to the class: demonstrate that the logic circuit below implements three bit flip encoding.



Reminder: CNOT gate



$|00\rangle \to |00\rangle$

$|01\rangle \to |01\rangle$

$|10\rangle \to |11\rangle$

$|11\rangle \to |10\rangle$

(flip control qubit if target qubit is |1>)

$$|\psi_0\rangle = \alpha|\underbrace{00}0\rangle + \beta|\underbrace{10}0\rangle$$

CNOT: flip $10 \to 11$

$$|\psi_1\rangle = \alpha|0\underbrace{00}\rangle + \beta|1\underbrace{10}\rangle$$

second CNOT: flip $10 \to 11$

$$\boxed{|\psi_2\rangle = \alpha|000\rangle + \beta|111\rangle}$$

How does this three-bit flip code works?

Suppose bit flip occurs on one or less qubits. Two-step error correcting procedure is used. Note: each of the qubits is passed through the independent bit-flip channel.
Error syndrome diagnosis (or error-detection): we perform the measurement telling us which error, if any, occurred on the quantum state. The measurement result is called the error syndrome.

For the bit-flip channel, there are four error syndromes corresponding to four projection operators:

$$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111| \qquad \text{no error}$$

$$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011| \qquad \text{bit flip on qubit \#1}$$

$$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101| \qquad \text{bit flip on qubit \#2}$$

$$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110| \qquad \text{bil flip on qubit \#3}$$

Suppose the bit flip occurred on qubit #1.
Then, the corrupted state is

$$|\psi\rangle = \alpha|100\rangle + \beta|011\rangle.$$

Note that $\langle\psi|P_1|\psi\rangle = 1$

Proof: $P_1|\psi\rangle = [\,|100\rangle\langle 100| + |011\rangle\langle 011|\,](\alpha|100\rangle + \beta|011\rangle)$

$$= \alpha|100\rangle + \beta|011\rangle$$

$$\langle\psi|P_1|\psi\rangle = (\langle 100|\alpha^* + \langle 011|\beta^*)(\alpha|100\rangle + \beta|011\rangle)$$

$$= \alpha^*\alpha + \beta^*\beta = 1$$

State of the system after measurement is $\dfrac{P_m|\psi\rangle}{\sqrt{p(m)}}$; $p(m) = \langle\psi|P_m|\psi\rangle$

$m = 1 \Rightarrow$

It is the same as before the measurement $P_1|\psi\rangle = |\psi\rangle = \alpha|100\rangle + \beta|011\rangle.$

Therefore, the result of the measurement is 1 with 100% probability and the measurement **does not change the state**. We now have the information about which error occurred but we learned nothing about α or β.

Recovery: the value of the error syndrome is used to recover the initial state. The four possible error syndromes and the recovery procedures in each case are:

0 (no error) - do nothing
1(bit flip on the first qubit) - flip the first qubit again (use X gate)
2 (bit flip on the second qubit) - flip the second qubit again
3 (bit flip on the third qubit) - flip the third qubit again

Exercise for the class:

What is the corrupted state $|\psi\rangle$ if the bit flip occurred on qubit #3?

Show that p(3)=1 and $P_3|\psi\rangle = |\psi\rangle$, so that the detection of this error by measurement will not change the state of the system.

$$|\psi\rangle = \alpha|001\rangle + \beta|110\rangle$$

$$P_3|\psi\rangle = |001\rangle\langle 001|\left(\alpha|001\rangle + \beta|110\rangle\right) + |110\rangle\langle 110|\left(\alpha|001\rangle + \beta|110\rangle\right) = \alpha|001\rangle + \beta|110\rangle$$

$$p(3) = \langle\psi|P_3|\psi\rangle = \left(\langle 001|\alpha^* + \langle 110|\beta^*\right)$$

$$\left(|001\rangle\alpha + |110\rangle\beta\right) = \alpha^*\alpha + \beta^*\beta = 1$$