Angular momentum: summary

If $[L_x, L_y] = i\hbar L_z$
$[L_y, L_z] = i\hbar L_x$
$[L_z, L_x] = i\hbar L_y$ then

Eigenfunctions f_ℓ^m of L^2 and L_z are labeled by m and l:

$$L^2 f_\ell^m = \underbrace{\hbar^2 \ell(\ell+1)}_{\text{eigenvalue of } L^2} f_\ell^m, \qquad L_z f_\ell^m = \underbrace{\hbar m}_{\text{eigenvalue of } L_z} f_\ell^m.$$

$\ell = 0, 1, 2, \ldots$ (only integer values for orbital angular momentum)

For a given value of l, there are $2l+1$ values of m: $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$.

Generally, half-integer values are also allowed (but not for orbital angular moment).

Elementary particles carry intrinsic angular momentum **S** in addition to **L**.
Spin of elementary particles has nothing to do with rotation, does not depend on coordinates θ and ϕ, and is purely a quantum mechanical phenomena.

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right].$$

Spin $1/2$

$s=\frac{1}{2}$, therefore $m=\pm\frac{1}{2}$ and there are two eigenstates $|s\,m\rangle = |\frac{1}{2}\,\frac{1}{2}\rangle$,

$|s\,m\rangle = |\frac{1}{2}\,(-\frac{1}{2})\rangle$.

We will call them spin up $\uparrow\ |\frac{1}{2}\ \underbrace{\tfrac{1}{2}}_{}\rangle$ ($m=\frac{1}{2}$) and spin down $\downarrow\ |\frac{1}{2}\ -\frac{1}{2}\rangle$ ($m=-\frac{1}{2}$).

↑ I will omit ( ).

Taking these eigenstates to be basis vectors, we can express any spin state of a particle with spin $1/2$ as:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

χ_+ represents spin up ↑

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

χ_- represents spin down ↓

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

All our spin operators are 2x2 matrixes for spin $1/2$, which we can find out from how they act on our basis set states χ_+ and χ_-.

Raising and lowering operators:

$$S_\pm = S_x \pm i S_y$$

$$\boxed{S_\pm |s\,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s\,(m\pm 1)\rangle}$$

Class exercise

The electron in a hydrogen atom occupies the combined spin an position state:

$$\psi = \underset{\substack{n=2 \\ \ell=1}}{R_{21}} \left(\sqrt{1/3}\; \underbrace{Y_1^0}_{\substack{\ell=1 \\ m_\ell=0}} \underbrace{\chi_+}_{\substack{s=\frac{1}{2} \\ m_s=\frac{1}{2}}} + \sqrt{2/3}\; \underbrace{Y_1^1}_{\substack{\ell=1 \\ m_\ell=1}} \underbrace{\chi_-}_{\substack{s=\frac{1}{2} \\ m_s=-\frac{1}{2}}} \right)$$

Note that $m_\ell + m_s = \frac{1}{2}$ in both cases

(a) If you measure the orbital angular momentum squared L^2, what values might your get and what is the probability of each?

$\ell = 1 \qquad L^2 \psi = \ell(\ell+1)\hbar^2 \psi \quad \Rightarrow$

You get $\hbar^2 \ell(\ell+1) = 2\hbar^2$ with probability P=1 (100%).

(b) Same for z component of the orbital angular momentum, L_z.

$L_z \psi = \hbar m_\ell \psi$

Possible values of m_ℓ: $m_\ell = 0$ or 1.

$P = \left(\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{3}$ for $m_\ell = 0$

$P = \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$ for $m_\ell = 1$.

(c) Same for the spin angular momentum squared S^2.

$S^2 \psi = \hbar^2 s(s+1)\psi \qquad s = \frac{1}{2} \Rightarrow$ You get $\frac{3}{4}\hbar^2$ with $P=1$.

(d) Same for z component of the spin angular momentum S_z.

$S_z \psi = \hbar m_s \psi \qquad m_s = \frac{1}{2}$ and $-\frac{1}{2}$

$P = 1/3$ for $m_s = \frac{1}{2}$; $P = 2/3$ for $m_s = -\frac{1}{2}$

Class exercise

Construct the spin matrix S_z for a particle of spin 1.

Solution Spin $s=1$

Hint: for a given value of s, there are 2s+1 values of m, so there are three eigenstates:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$s=1$ $m=1$ $\qquad$ $s=1$ $m=0$ $\qquad$ $s=1$ $m=-1$

For the operator Sz:

$$S_z |sm\rangle = m\hbar |sm\rangle \Rightarrow$$

$$S_z \chi_+ = \hbar \chi_+ \qquad S_z \chi_- = -\hbar \chi_- \qquad S_z \chi_0 = 0$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & k & e \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a \\ d \\ g \end{pmatrix} = \begin{pmatrix} \hbar \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} a=\hbar \\ d=0 \\ g=0 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & k & l \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} b \\ e \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} b=0 \\ e=0 \\ k=0 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & k & e \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} c \\ f \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\hbar \end{pmatrix} \qquad \begin{matrix} c=0 \\ f=0 \\ l=-\hbar \end{matrix} \Rightarrow$$

$$\boxed{S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

Addition of angular momenta

Let's go back to ground state of hydrogen: it has one proton with spin $1/2$ and one electron with spin $1/2$ (orbital angular momentum is zero). What is the total angular momentum $\vec{S}$ of the hydrogen atom?

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Total spin

Electron's spin, acts only on electron's spin states

Proton's spin, acts only on proton's spin states

$$S_z \chi_1 \chi_2 = \left(S_z^{(1)} + S_z^{(2)} \right) \chi_1 \chi_2 =$$

electron's spin state

proton's spin state

$$= \left(S_z^{(1)} \chi_1 \right) \chi_2 + \chi_1 \left(S_z^{(2)} \chi_2 \right) = \hbar m_1 \chi_1 \chi_2 + \hbar m_2 \chi_1 \chi_2$$

$$= \hbar \left(m_1 + m_2 \right) \chi_1 \chi_2$$

Therefore, the z components just add together and quantum number m for the composite system is simply

$$\boxed{m = m_1 + m_2} \, .$$

There are four possible combinations:

$m_1 = \frac{1}{2}$	$m_2 = \frac{1}{2}$	$\uparrow\uparrow$	$m = 1$
$m_1 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$	$\uparrow\downarrow$	$m = 0$
$m_1 = -\frac{1}{2}$	$m_2 = -\frac{1}{2}$	$\downarrow\uparrow$	$m = 0$
$m_1 = -\frac{1}{2}$	$m_2 = -\frac{1}{2}$	$\downarrow\downarrow$	$m = -1$

(first arrow corresponds to the electron spin and second arrow corresponds to the nuclear spin) .

Well, it appears that we have an extra state!

Let's apply lowering operator to state $\uparrow\uparrow$ to sort this out :

$$S_- |s\, m\rangle = \hbar \sqrt{s(s+1) - m(m-1)}\, |s\;\, m-1\rangle$$

$$S_-^{(1)} \uparrow = S_-^{(1)} \left|\tfrac{1}{2}\; \tfrac{1}{2}\right\rangle = \hbar \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}\, \left|\tfrac{1}{2}\; -\tfrac{1}{2}\right\rangle = \hbar \downarrow$$

state with $s = 1/2,\ m = 1/2$

state with $s = \frac{1}{2}\ \ m = -\frac{1}{2}$

$$S_-(\uparrow\uparrow) = \left(S_-^{(1)} + S_-^{(2)}\right)\uparrow\uparrow$$

$$= \left(S_-^{(1)}\uparrow\right)\uparrow + \uparrow\left(S_-^{(2)}\uparrow\right) = \hbar\downarrow\uparrow + \hbar\uparrow\downarrow = \hbar(\downarrow\uparrow + \uparrow\downarrow)$$

(Note: normalization is not preserved here).

So we can sort out four states as follows:

Three states $|s\; m\rangle$ with spin s = 1, m = 1, 0, -1 :

$$\left.\begin{cases} |1\,1\rangle = \uparrow\uparrow \\ |1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,{-1}\rangle = \downarrow\downarrow \end{cases}\right\} S=1$$

This is called a **triplet** configuration.

and one state with spin s = 0, m = 0:

$$\left\{ |0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\} \quad S=0$$

This is called a **singlet** configuration.

Summary: Combination of two spin $1/2$ particles can carry a total spin of s =1 or s = 0, depending on whether they occupy the triplet or singlet configuration.

Addition of angular momenta

Ground state of hydrogen: it has one proton with spin $1/2$ and one electron with spin $1/2$ (orbital angular momentum is zero). What is the total angular momentum $\vec{S}$ of the hydrogen atom?

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Total spin

Electron's spin, acts only on electron's spin states

Proton's spin, acts only on proton's spin states

The z components just add together and quantum number m for the composite system is simply

$$\boxed{m = m_1 + m_2} \, .$$

In general, it you combine any angular momentum j_1 and j_2 you get every value of angular momentum from $|j_1 - j_2|$ to $j_1 + j_2$ in integer steps:

$$j = |j_1 - j_2|, \ldots, (j_1 + j_2)$$

It does not matter if it is orbital angular momentum or spin.

Example:

$$j_1 = \frac{3}{2} \qquad j_2 = 3 \quad \Rightarrow$$

$$|j_1 - j_2| = \left| \frac{3}{2} - 3 \right| = \frac{3}{2}$$

$$j_1 + j_2 = \frac{3}{2} + 3 = \frac{9}{2}$$

Total angular momentum j can be $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, and $\frac{9}{2}$.

Problem

Quarks carry spin $1/2$. Two quarks (or actually a quark and an antiquark) bind together to make a meson (such as pion or kaon). Three quarks bind together to make a barion (such as proton or neutron). Assume all quarks are in the ground state so the orbital angular momentum is zero).

(1) What spins are possible for mesons?

(2) What spins are possible for baryons?

Solution:

(1) $s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2} \Rightarrow \boxed{s = 0 \text{ or } 1.}$

(2) $\underbrace{s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2}}_{\text{Add these first}} \quad s_3 = \frac{1}{2}$

$s_{12} = 0 \text{ or } 1$

Now add the third spin: $s_{12} = 0 \quad s_3 = \frac{1}{2} \Rightarrow s = \frac{1}{2}$

$s_{12} = 1 \quad s_3 = \frac{1}{2} \Rightarrow s = \left|\frac{1}{2} - 1\right|, \ldots \left(\frac{1}{2} + 1\right) \Rightarrow$

$s = \frac{1}{2} \text{ or } \frac{3}{2}$

$$\boxed{s = \frac{1}{2} \text{ or } \frac{3}{2}}$$