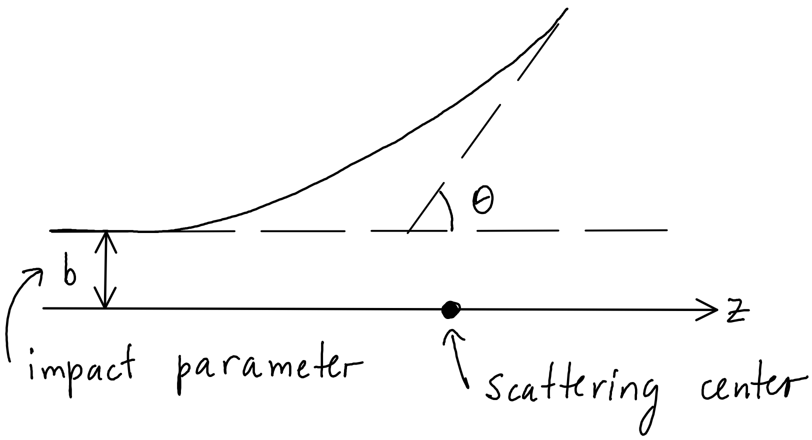
Lecture 18

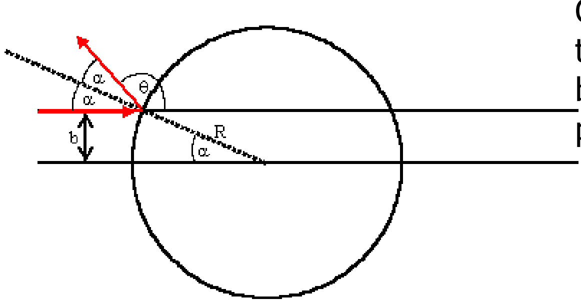
Scattering

Classical scattering theory

Problem: given the scattering parameter b, calculate the scattering angle θ

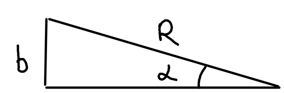


Example: elastic hard-sphere scattering

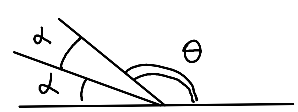


Our target is a billiard ball of radius R, the incident particle is a ball that bounces elastically. The impact parameter is

$$b = R \sin \alpha$$



The scattering angle is



$$\theta = \pi - 2\alpha$$

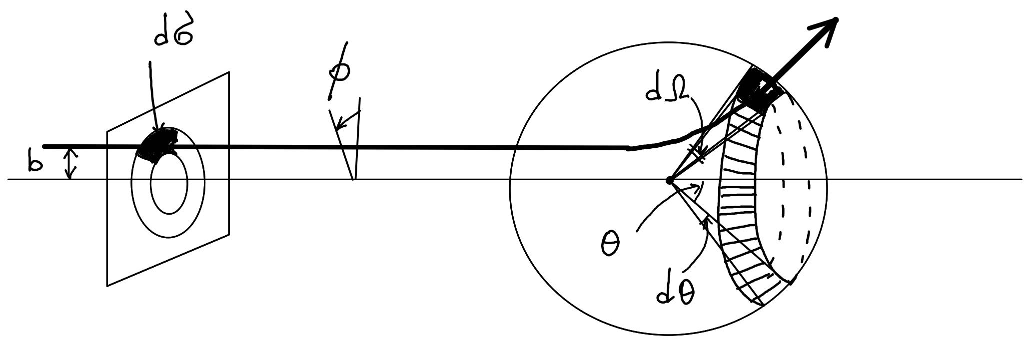
$$\alpha = \frac{1}{2}(\pi - \theta)$$

$$b = R \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R \cos\frac{\theta}{2}$$

Therefore, the scattering angle is:

$$\theta = \begin{cases} 2\cos^{-1}(b/R) & \text{if} \quad b \le R \\ 0 & \text{if} \quad b \geq R \end{cases}$$

Scattering



More general problem: particles incident within an infinitesimal patch of cross-sectional area $d\sigma$ are scattering into an infinitesimal solid angle $d\Omega$.
The quantity

$$\boxed{D(\theta) = \frac{d\sigma}{d\Omega}}$$

is called the differential (scattering) cross-section.

$$d\sigma = b\, db\, d\phi$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

$$\boxed{D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

Total cross section is defined as the integral of D(θ) over all solid angles:

$$\boxed{\sigma \equiv \int D(\theta)\, d\Omega}$$

Class exercise: Find the differential and total cross-sections for hard-sphere scattering.

$$b = R\cos\frac{\theta}{2} \qquad \frac{db}{d\theta} = -\frac{1}{2}R\sin\frac{\theta}{2}$$

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{R\cos\frac{\theta}{2}}{\sin\theta}\,\frac{1}{2}R\sin\frac{\theta}{2} = \underline{\underline{\frac{R^2}{4}}}$$

$$\sigma = (R^2/4)\int d\Omega = \pi R^2$$

If we have a beam of incident particles, with uniform intensity (luminosity)

$\mathcal{L} \equiv$ number of incident particles per unit area per unit time $\Rightarrow$

The member of particles entering $d\sigma$ per unit time is

$$dN = \mathcal{L}\, d\sigma = \mathcal{L}\, D(\theta)\, d\Omega \quad \Rightarrow$$

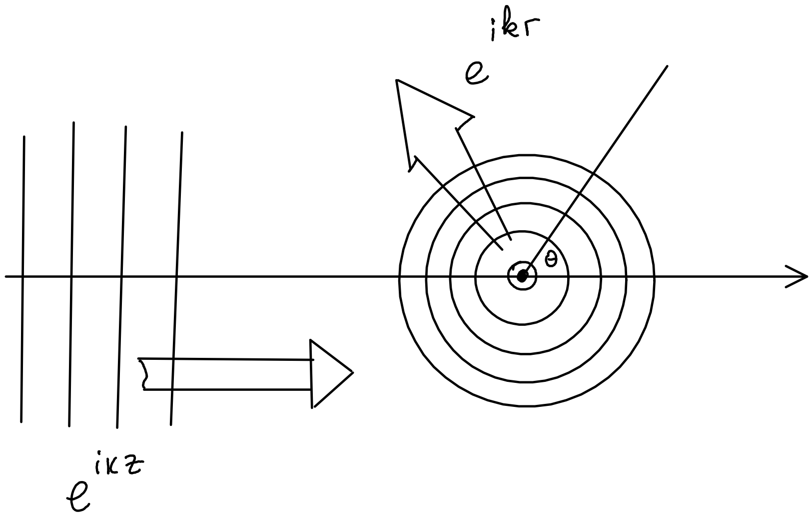
$$\boxed{D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}}$$

Quantum scattering theory

Our problem: incident plane wave

$$\psi(z) = A e^{ikz}$$

traveling in Z direction encounters a scattering potential that produces outgoing spherical wave:



Therefore, the solutions of the Schrödinger equation have the general form:

$$\psi(r,\theta) \approx A\left\{ \underbrace{e^{ikz}}_{\text{plane wave}} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{\text{spherical wave}} \right\} \quad \text{for large } r$$

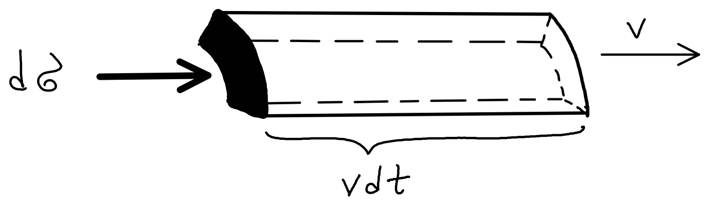
$$k = \frac{\sqrt{2mE}}{\hbar}$$

E – energy of the incident particles

The quantity $f(\theta)$ is called scattering amplitude.

It is the probability of scattering in a given direction θ.

How is it connected to the differential cross-section?



$$dV = v\,dt\,d\sigma$$

Volume dV of incident beam (see above) passes through area $d\sigma$ in time dt.
The probability that the particle with speed v passes through this area $d\sigma$ is

$$dP = |\underbrace{\psi_{incident}}_{= A e^{ikz}}|^2\, dV = |A|^2\, v\, dt\, d\sigma$$

This must be equal to the probability that the particle scatters into the solid angle $d\Omega$:

$$dP = |\underbrace{\psi_{scattered}}_{A \frac{e^{ikz}}{r} f(\theta)}|^2\, dV = \frac{|A|^2 |f|^2}{r^2} (v\,dt)\, r^2\, d\Omega$$

Therefore $\quad |A|^2 v\, dt\, d\sigma = \dfrac{|A|^2 |f(\theta)|^2}{r^2} (v\,dt)\, r^2\, d\Omega$

$$d\sigma = |f(\theta)|^2\, d\Omega \quad \Rightarrow$$

$$\boxed{D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2}$$

Therefore, to solve the scattering problem, we need to calculate the scattering amplitude $f(\theta)$.

Partial wave analysis

Our potential is spherically symmetric $\Longrightarrow$

The solutions of the Schrödinger equation are

$$\psi(r,\theta,\phi) = R(r)\, Y_\ell^m(\theta,\phi)$$

spherical harmonic

$u(r) = rR(r)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

(radial Schrödinger equation)

"centrifugal contribution"

$\boxed{\text{Very large } r}$ $\longrightarrow$ $V \to 0$

and "centrifugal contribution" is negligible

Radial equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} \approx Eu$$

$$\frac{d^2u}{dr^2} \approx -k^2 u$$

The general solution is

$$u = Ce^{ikr} + De^{-ikr}$$

outgoing spherical wave

incoming spherical wave $\Rightarrow$ $D = 0$

$$\boxed{R = \frac{Ce^{ikr}}{r}}$$