

Lecture 16

WKB approximation

Summary

"Classical" region

$$\psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x) dx} ; \quad p(x) = \sqrt{2m(E - V(x))}$$

General solution is the combination of these two.

If $E < V$, then p is imaginary but we can still write

$$\psi(x) \approx \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar} \int |p(x)| dx}$$

for non-classical region.

Transmission probability

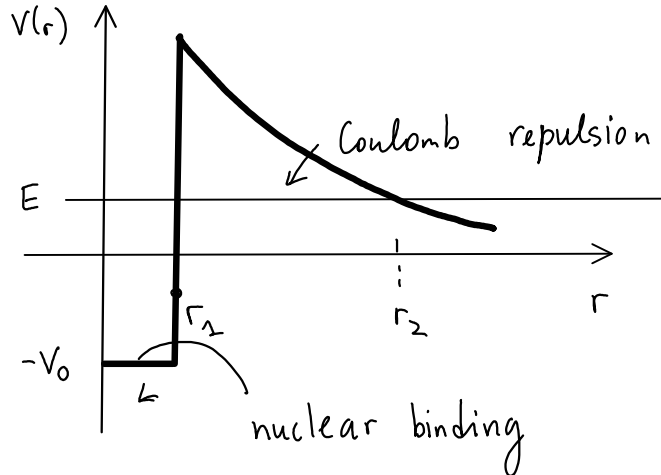
$$\frac{|F|}{|A|} = e^{-\frac{1}{\hbar} \int_0^a |p(x')| dx'}$$

$$T \approx e^{-2\gamma} \quad \gamma = \frac{1}{\hbar} \int_0^a |p(x)| dx$$

Example: Gamow's theory of alpha decay

Simulation: http://phet.colorado.edu/simulations/sims.php?sim=Alpha_Decay

Alpha decay is spontaneous emission of an alpha-particle (two protons and two neutrons) by radioactive nuclei.



If alpha -particle (charge $2e$) escapes the nuclear binding force it is repelled by the leftover nuclei with charge Ze .

E: energy of the emitted alpha-particle

$$\underbrace{\frac{1}{4\pi\epsilon_0} \frac{2e \cdot Ze}{r_2}} = E$$

V(r) for Coulomb repulsion

We now apply WKB approximation. The non-classical region is from r_1 to r_2 .

$$\gamma \equiv \frac{1}{\hbar} \int_0^a |p(x)| dx$$

$$\gamma = \frac{1}{\hbar} \int_{r_1}^{r_2} \sqrt{2m \left(\frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r} - E \right)} dr = \frac{1}{\hbar} \int_{r_1}^{r_2} \sqrt{2m \left(E \frac{r_2}{r} - E \right)} dr$$

$$= \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} \sqrt{\frac{r_2}{r} - 1} dr$$

substitute
 $r \equiv r_2 \sin^2 u$
to do integral

$$\gamma = \frac{\sqrt{2mE}}{\hbar} \left[r_2 \left(\frac{\pi}{2} - \frac{1}{\sin \sqrt{\frac{r_1}{r_2}}} \right) - \sqrt{r_1(r_2 - r_1)} \right]$$

$$r_1 \ll r_2 \Rightarrow \sin \epsilon \approx \epsilon$$

$$\gamma \approx \frac{\sqrt{2mE}}{\hbar} \left[\frac{\pi}{2} r_2 - 2\sqrt{r_1 r_2} \right] = \frac{\sqrt{2mE}}{\hbar} \frac{\pi}{2} \frac{2e^2 z}{4\pi\epsilon_0} \frac{1}{E}$$

$$r_2 = \frac{2e^2 z}{4\pi\epsilon_0} \frac{1}{E}$$

$$- \frac{\sqrt{2mE}}{\hbar} 2\sqrt{r_1} \frac{1}{\sqrt{E}} \sqrt{\frac{2e^2 z}{4\pi\epsilon_0}}$$

$$\gamma \approx \frac{z}{\sqrt{E}}$$

$$\frac{\sqrt{2m}}{\hbar} \pi \frac{e^2}{4\pi\epsilon_0}$$

$$K_1 = 1.980 \text{ MeV}^{1/2}$$

$$- \sqrt{2r_1}$$

$$\left(\frac{e^2}{4\pi\epsilon_0} \right)^{1/2} \frac{4\sqrt{m}}{\hbar}$$

$$K_2 = 1.485 \text{ fm}^{-1/2}$$

$$\gamma = K_1 \frac{z}{\sqrt{E}} - K_2 \sqrt{2r_1}$$

Probability for alpha-particle to escape at each "collision" with the "wall" is $e^{-2\gamma}$.

Average time between collision with the "wall" is $2r_1/v$

where v is its velocity. Therefore, frequency of the collisions is $v/2r_1$.

The probability of emission per unit time is $\frac{v}{2r_1} e^{-2\gamma}$.

The lifetime of the parent nucleus is $\tau = \frac{2r_1}{v} e^{2\gamma}$

This formula gives correct energy dependence (logarithm of τ vs. $1/\sqrt{E}$ is a straight line).

$$E = m_p c^2 - m_d c^2 - m_\alpha c^2$$

↑ mass of the parent nucleus
↑ mass of the daughter nucleus
↑ mass of the alpha particle

energy of the emitted particle

To estimate v : $E = \frac{1}{2} m_\alpha v^2$.

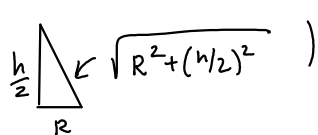
Problem

About how long would it take a can of beer at room temperature to topple spontaneously, as a result of quantum tunneling?

Solution

We are going to treat the can as uniform cylinder of mass m , radius R , and length h . As the can tips, let x to be the height of the center above equilibrium position $h/2$. The potential energy is mgx , and it topples when x reaches the critical value

$$x_0 = \sqrt{R^2 + (h/2)^2} - h/2$$

(half the diagonal )

We need to calculate tunneling probability for $E=0$. We can use equation

$$\gamma = \frac{1}{\hbar} \int_0^{x_0} \sqrt{2m(V-E)} dx \quad ; \quad V(x) = mgx \quad ; \quad E = 0$$

$$\gamma = \frac{1}{\hbar} \int_0^{x_0} \sqrt{mg} \sqrt{x} dx = \frac{m}{\hbar} \sqrt{2g} \frac{2}{3} x^{3/2} \Big|_0^{x_0} = \frac{2m}{3\hbar} \sqrt{2g} x_0^{3/2}$$

We estimate:

$$h = 10 \text{ cm}; \quad R = 3 \text{ cm}; \quad m = 300 \text{ gm}; \quad g = 9.8 \text{ m/s}^2$$

$$x_0 = \sqrt{9 + 25} - 5 = 0.83 \text{ cm};$$

$$\gamma = \frac{2 \cdot 0.3}{3 \times 1.05 \times 10^{-34}} \sqrt{2 \cdot 9.8} (0.0083)^{3/2} = 6.4 \times 10^{30}$$

Frequency of "attempts": $f = \frac{v}{2R}$. Time is given by

$$\tau = \frac{2R}{v} e^{2\gamma}$$

as we derived for alpha decay.

We can also derive it as follows: we want product of the number of attempts ($f\tau$) and probability of toppling at each attempt T to be 1:

$$(f\tau) T = \tau \frac{v}{2R} e^{-2\gamma} = 1 \Rightarrow \tau = \frac{2R}{v} e^{2\gamma}$$

We can estimate thermal velocity from

$$\frac{1}{2} m v^2 = \frac{1}{2} k_B T$$

$$T = 300 \text{ K}$$

here, T is temperature,
not tunneling probability.

$$v = \sqrt{k_B T / m}$$

$$\tau = 2 R \sqrt{\frac{m}{k_B T}} e^{2\gamma} = 2 \cdot (0.03) \sqrt{\frac{0.3}{(1.4 \times 10^{-23}) (300)}} \times$$

$$\times e^{12.8 \times 10^{30}} \approx 1.6 \times 10^{5.6 \times 10^{30}} \text{ years.}$$

Connection formula

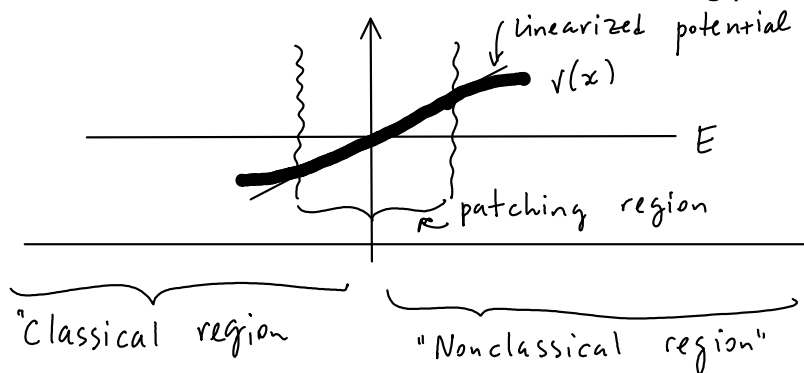
So far, we assumed that the "walls" of the potential are vertical. In this case, boundary condition is trivial. Our results are also valid even if the edges are not so abrupt. However, we need to consider what actually happens at the turning point, where "classical" region connects with "non classical" region. In this case WKB approximation breaks down.

We consider bound state problem in this lecture. It is straightforward to extend this solution to scattering problem.

We shift the axes so the right-hand turning point happens at $x=0$. In the WKB approximation, the wave function at the left and right of the turning point are:

$$(1) \quad \psi(x) = \begin{cases} \frac{1}{\sqrt{p(x)}} \left\{ B e^{\frac{i}{\hbar} \int_x^0 p(x') dx'} + C e^{-\frac{i}{\hbar} \int_x^0 p(x') dx'} \right. & x < 0 \\ \frac{1}{\sqrt{|p(x)|}} D e^{\frac{1}{\hbar} \int_0^x |p(x')| dx'} & x > 0 \end{cases}$$

The positive exponent solution is excluded for $x > 0$ since it blows up at infinity assuming that $V(x)$ is greater than E for all $x > 0$. Since we can not use WKB approximation at turning point (since $p \rightarrow 0$ and $\psi \rightarrow \infty$) we will splice the WKB solutions together using the patching function that describes the wave function at the turning point.



Since we only need the wave function near the origin we can approximate a potential there by a straight line. To get the wave function, we need so solve the Schrödinger equation with this potential.

$$V(x) \cong E + v'(0)x$$

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_p}{dx^2} + [E + v'(0)x] \psi_p = E \psi_p$$

$$\frac{d^2 \psi_p}{dx^2} = \alpha^3 x \psi_p$$

$$\alpha \equiv \left[\frac{2m}{\hbar^2} V'(0) \right]^{1/3}$$

Substitute: $z \equiv \alpha x \Rightarrow$

$$\frac{d^2 \psi_p}{dz^2} = z \psi_p \leftarrow \text{Airy's equation}$$

The solutions are Airy functions. There are two linearly independent Airy functions $Ai(z)$ and $Bi(z)$.

Asymptotic Forms:

$$\left. \begin{aligned} Ai(z) &\sim \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3} z^{3/2}} \\ Bi(z) &\sim \frac{1}{\sqrt{\pi} z^{1/4}} e^{\frac{2}{3} z^{3/2}} \end{aligned} \right\} z \gg 0$$

$$Ai(z) \sim \frac{1}{\sqrt{\pi} (-z)^{1/4}} \sin \left[\frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right]$$

$$Bi(z) \sim \frac{1}{\sqrt{\pi} (-z)^{1/4}} \cos \left[\frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right]$$

To get connecting formulas, we match our actual functions (1) with asymptotic forms of Airy functions. The resulting formulas are:

$$\psi(x) \cong \begin{cases} \frac{2D}{\sqrt{p(x)}} \sin \left[\frac{1}{\hbar} \int_x^{x_2} p(x') dx' + \frac{\pi}{4} \right] & x < x_2 \\ \frac{D}{\sqrt{|p(x)|}} \exp \left[-\frac{1}{\hbar} \int_{x_2}^x |p(x')| dx' \right] & x > x_2 \end{cases}$$

x_2 is turning point