Lecture 15

WKB approximation

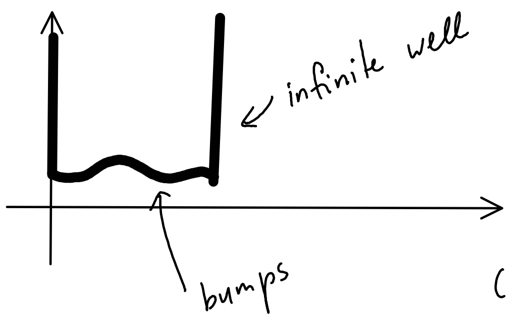
Summary ("classical" region)

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar}\int p(x)\,dx} \; ; \qquad p(x) = \sqrt{2m(E - V(x))}$$

General solution is the combination of these two.

Example

Potential with two vertical walls



$$V(x) = \begin{cases} \text{some function} & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

$E < V$

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)}\right] \tag{1}$$

We can also write (1) as

$$\psi(x) = \frac{1}{\sqrt{p(x)}}\left[C_1 \sin\phi(x) + C_2 \cos\phi(x)\right]$$

$$\phi(x) = \frac{1}{\hbar}\int_0^x p(x')\,dx'$$

$\psi(x) = 0$ when $x = 0 \Rightarrow \psi(0) = 0$

$$\psi(0) = \frac{1}{\sqrt{p(0)}}\left(C_1 \overset{0}{\cancel{\sin 0}} + C_2 \overset{=1}{\cancel{\cos 0}}\right) = 0 \implies C_2 = 0$$

$\psi(x) = 0$ when $x = a \Rightarrow \psi(a) = 0$

$$\psi(a) = \frac{1}{\sqrt{p(a)}} C_1 \sin\phi(a) = 0 \Rightarrow \sin[\phi(a)] = 0 \Rightarrow$$

$$\boxed{\phi(a) = \pi n, \quad n = 1, 2, 3, \ldots}$$

$$\phi(a) = \pi n \quad \Rightarrow \quad \phi(a) = \frac{1}{\hbar}\int_0^a p(x)\,dx = \pi n$$

$$\boxed{\int_0^a p(x)\,dx = \pi n \hbar \ ; \quad n = 1, 2, 3 \ldots}$$

This condition determines the approximate allowed energies.

If V(x)=0, then

$$p(x) = \sqrt{2mE}$$

$$\int_0^a p(x)\,dx = \int_0^a \sqrt{2mE_n}\,dx = \sqrt{2mE_n}\,a = \pi n \hbar$$

$$2mE_n a^2 = \pi^2 n^2 \hbar^2$$

$$\boxed{E_n = \frac{\pi^2 n^2 \hbar^2}{2ma^2}}$$

Exact answer, since V(x) constant, i.e. we could drop A" with no consequences.

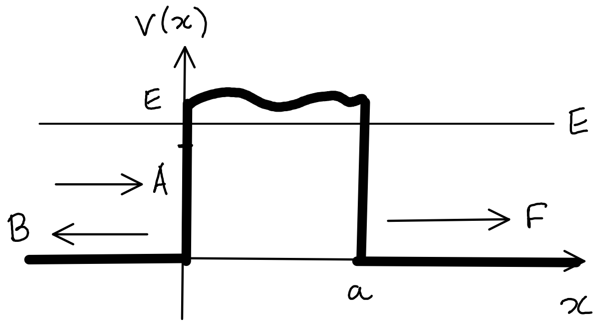
Tunneling

If E<V, then p is imaginary but we can still write

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar}\int |p(x)|\,dx}$$

for non-classical region.

Example:



A: incident amplitude
B: reflected amplitude
F: transmitted amplitude

To the left of the barrier: $\psi(x) = Ae^{ikx} + Be^{-ikx}$ $\quad k = \sqrt{2mE}/\hbar$

To the right of the barrier: $\psi(x) = Fe^{ikx}$

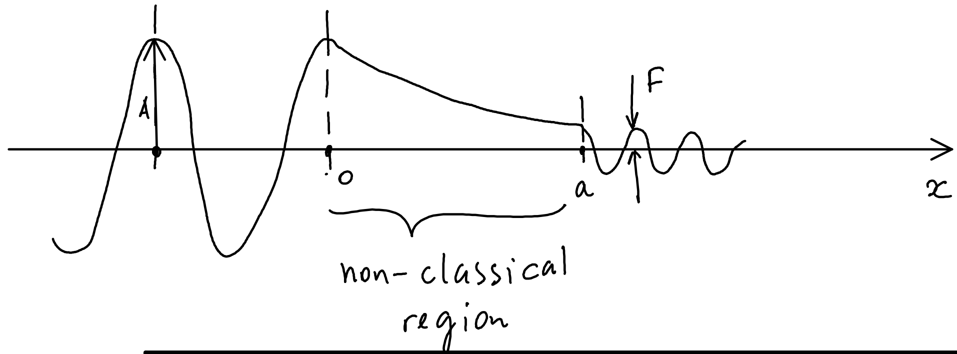
Transmission probability: $T = \frac{|F|^2}{|A|^2}$

Tunneling (nonclassical) region: $(0 < x < a)$

WKB approximation gives

$$\psi(x) \simeq \frac{C}{\sqrt{|p(x)|}} e^{\frac{1}{\hbar}\int_0^x |p(x')|\,dx'} + \frac{D}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_0^x |p(x')|\,dx'}$$

If the barrier is very high (or very wide), the tunneling probability is small. Then, the coefficient in the exponentially increasing term must be small. The relative amplitude of the incident and transmitted waves are determined by the decrease of the exponential over the nonclassical region.



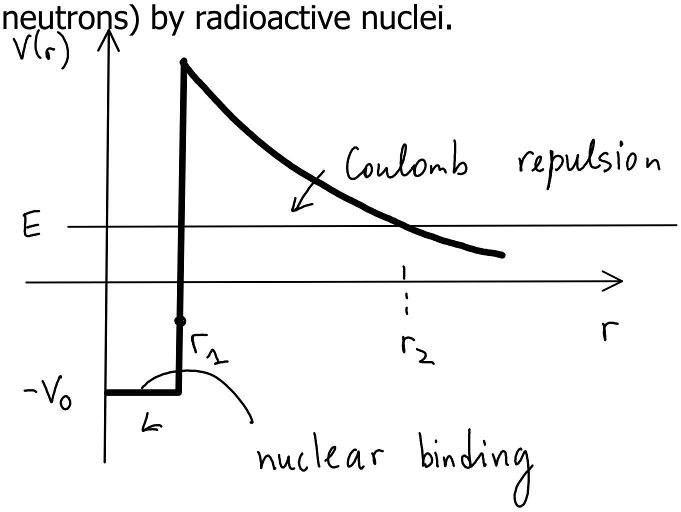
$$\frac{|F|}{|A|} = e^{-\frac{1}{\hbar}\int_0^a |p(x')|\,dx'}$$

$$T \cong e^{-2\gamma} \qquad \gamma = \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$

Example: Gamow's theory of alpha decay

Simulation: http://phet.colorado.edu/simulations/sims.php?sim=Alpha_Decay

Alpha decay is spontaneous emission of an alpha-particle (two protons and two neutrons) by radioactive nuclei.



If alpha -particle (charge 2e) escapes the nuclear binding force it is repelled by the leftover nuclei with charge Ze.

E:energy of the emitted alpha-particle

$$\underbrace{\frac{1}{4\pi\epsilon_0}\frac{2e\,Ze}{r_2}}_{\text{V(r) for Coulomb repulsion}} = E$$

We now apply WKB approximation. The non-classical region is from r_1 to r_2.

$$\gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2}\sqrt{2m\left(\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r} - E\right)}\,dr = \frac{1}{\hbar}\int_{r_1}^{r_2}\sqrt{2m\left(E\frac{r_2}{r} - E\right)}\,dr$$

$$= \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2}\sqrt{\frac{r_2}{r} - 1}\,dr$$

substitute $r \equiv r_2 \sin^2 u$ to do integral

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\left[r_2\left(\frac{\pi}{2} - \frac{1}{\sin\sqrt{\frac{r_1}{r_2}}}\right) - \sqrt{r_1(r_2 - r_1)}\right]$$

$$r_1 \ll r_2 \quad \Rightarrow \quad \sin\epsilon \approx \epsilon$$

$$\gamma \simeq \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2}r_2 - 2\sqrt{r_1 r_2}\right] = \frac{\sqrt{2mE}}{\hbar}\frac{\pi}{\cancel{2}}\frac{\cancel{2}e^2 Z}{4\pi\epsilon_0}\frac{1}{E} - \frac{\sqrt{2m\cancel{E}}}{\hbar}2\sqrt{r_1}\frac{1}{\cancel{\sqrt{E}}}\sqrt{\frac{2e^2Z}{4\pi\epsilon_0}}$$

$$\boxed{r_2 = \frac{2e^2Z}{4\pi\epsilon_0}\frac{1}{E}}$$

$$\gamma \cong \frac{Z}{\sqrt{E}}\underbrace{\boxed{\frac{\sqrt{2m}}{\hbar}\pi\frac{e^2}{4\pi\epsilon_0}}}_{=\,K_1 = 1.980\ \text{MeV}^{1/2}} - \sqrt{Zr_1}\underbrace{\boxed{\left(\frac{e^2}{4\pi\epsilon_0}\right)^{1/2}\frac{4\sqrt{m}}{\hbar}}}_{=\,K_2 = 1.485\ \text{fm}^{-1/2}}$$

$$\boxed{\gamma = K_1\frac{Z}{\sqrt{E}} - K_2\sqrt{Zr_1}}$$

Probability for alpha-particle to escape at each "collision" with the "wall" is $e^{-2\gamma}$.

Average time between collision with the "wall" is $2r_1/v$

where v is its velocity. Therefore, frequency of the collisions is $v/2r_1$.

The probability of emission per unit time is $\frac{v}{2r_1} e^{-2\gamma}$.

The lifetime of the parent nucleus is

$$\tau = \frac{2r_1}{v} e^{2\gamma}$$

This formula gives correct energy dependence (logarithm of τ vs. $1/\sqrt{E}$ is a straight line).

$$E = m_p c^2 - m_d c^2 - m_\alpha c^2$$

energy of the emitted particle

mass of the parent nucleus

mass of the daughter nucleus

mass of the alpha particle

To estimate v: $E = \frac{1}{2} m_\alpha v^2$.