Lecture 8

Analytical method: review

Our mission: solve $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \quad (E.1)$$

Step 1: change variables $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\,x;\quad K\equiv\frac{2E}{\hbar\omega}$

Resulting equation: $$\frac{d^2\psi}{d\xi^2}=(\xi^2-K)\psi \quad \text{(E.2)}$$

Step2: Find out asymptotic behavior at $\xi\to\infty$ and separate the resulting function $e^{-\xi^2/2}$ out

$$\psi(\xi)=h(\xi)\,e^{-\xi^2/2}$$

Resulting equation: $$\frac{d^2h}{d\xi^2}-2\xi\frac{dh}{d\xi}+(K-1)h=0 \quad (E.3)$$

Step 3: look for solutions of this equation (E.3) in the form of the power series

$$h(\xi)=a_0+a_1\xi+a_2\xi^2+\ldots=\sum_{j=0}^{\infty}a_j\xi^j$$

Resulting equation: (recursion formula) $$a_{j+2}=\frac{2j+1-K}{(j+1)(j+2)}a_j \quad (E.4)$$

Step 4: Need to truncate $\sum_{j=0}^{\infty} a_j \xi^j$ sum somewhere to ensure that all solutions are normalizable

Power series must terminate, i.e. $a_{n+2} = 0$ for some $n = j_{max}$.

Resulting equation $K = 2n+1 \Rightarrow E_n = (n+\frac{1}{2})\hbar\omega$, $n=0,1,\ldots$

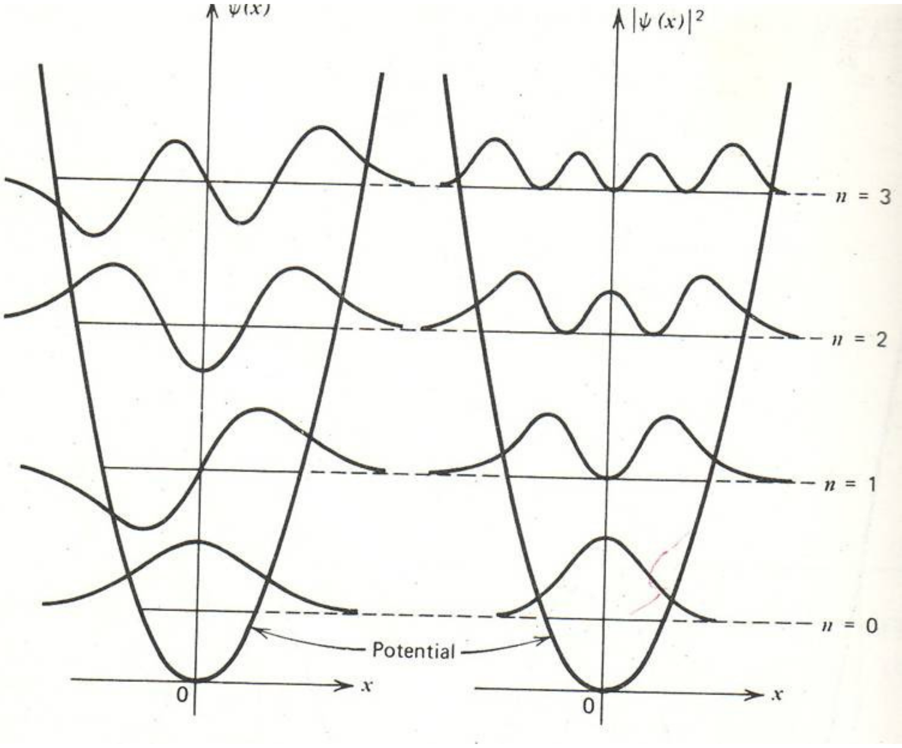
Step 5: Put all together and generate wave functions

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \underset{\substack{\uparrow \\ \text{Hermite polynomials}}}{H_n(\xi)} e^{-\xi^2/2}$$

Class exercise: find $\psi_0(\xi)$, $\psi_1(\xi)$, and $\psi_2(\xi)$ using (E.4).

See lecture 7 for solution to class exercise.

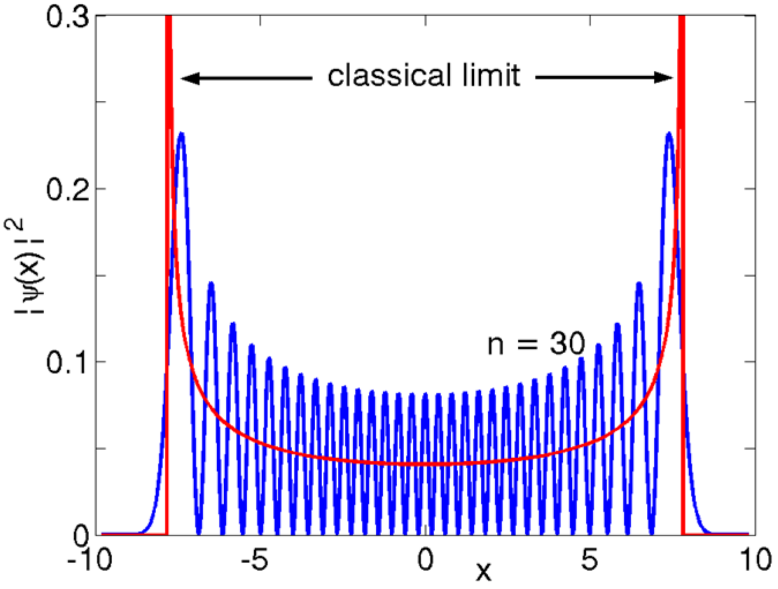
Of course, we get the same result as before using $a_{\pm}$ operators.



Note that the probability of finding the particle outside of classically allowed region is not zero. All wave functions extend beyond the potential energy curve representing the classically allowed maximum displacement of the oscillator.

Classically, the energy of the oscillator is $E = \frac{1}{2}ka^2 = \frac{1}{2}m\omega^2 a^2$, where a is the amplitude.

Only for large n we see some resemblance to classical case.



Computer simulation: http://www.falstad.com/mathphysics.html
1D Quantum mechanics applet
Harmonic oscillator

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \qquad V = \frac{1}{2}m\omega^2 x^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\langle x\rangle = 0 \qquad \langle x^2\rangle = \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}$$