

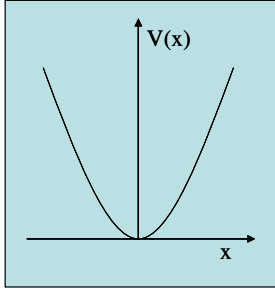
Lecture 6

The Harmonic Oscillator

Classical harmonic oscillator: mass m attached to a spring of force constant k .

Hooke's law:

$$F = -kx$$



$$F_x = m a_x$$

$$-kx = m \frac{d^2 x}{dt^2}$$

$$x(t) = A \sin(\omega t) + B \cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}}$$

The potential energy is $V = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2$

Quantum problem:

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

Solve Schrödinger equation with this potential.

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = E \psi$$

Two methods to solve it. We will start from algebraic method (trick solution).

Algebraic method

$$p = -i\hbar \frac{\partial}{\partial x} \Rightarrow$$

$$\frac{1}{2m} [p^2 + (m\omega x)^2] \psi = E\psi \quad \hat{H}\psi = E\psi$$

The basic idea is to factor the Hamiltonian

$$H = \frac{1}{2m} [p^2 + (m\omega x)^2]$$

For numbers, the solution is obvious $u^2 + v^2 = (iu + v)(-iu + v)$

We can not do it here since p and x are operators and xp is not the same as px .

Still, let's consider $a_+ = \frac{1}{\sqrt{2\hbar m\omega}} (-ip + m\omega x)$ and

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}} (ip + m\omega x)$$

↑
factor is for convenience

We can write a_+ and a_- together:

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp ip + m\omega x)$$

First, we calculate $a_- a_+$ and see what we will get:

$$\begin{aligned} a_- a_+ &= \frac{1}{2\hbar m\omega} (ip + m\omega x)(-ip + m\omega x) = \\ &= \frac{1}{2\hbar m\omega} (p^2 + (m\omega x)^2 + ipm\omega x - m\omega x ip) = \end{aligned}$$

↑ ↑ ↑ ↑ ↑ ↑
constants we can move around

$$a_- a_+ = \frac{1}{2\hbar m\omega} [p^2 + (m\omega x)^2 - \underbrace{im\omega(xp - px)}_{\text{L6. P3}}]$$

this is called a commutator

$$[x, p] = xp - px$$

In general, the commutator of any two operators is $[A, B] = AB - BA$.

Now, we need to calculate commutator $[x, p]$. To deal with operators, use test function $f(x)$ (any function).

$$\begin{aligned} [x, p] f(x) &= xp f(x) - px f(x) = \\ &= x(-i\hbar \frac{d}{dx})f + i\hbar \frac{d}{dx}(xf) = -i\hbar \cancel{x \frac{df}{dx}} + i\hbar \frac{dx}{dx} f \\ &\quad + i\hbar \cancel{x \frac{df}{dx}} = i\hbar f(x) \end{aligned}$$

So, $[x, p] f(x) = i\hbar f(x)$ for any $f(x)$.

Therefore,

$$\boxed{[x, p] \equiv i\hbar} \leftarrow \text{canonical commutation relation}$$

We now find $a_- a_+ = \frac{1}{2\hbar m\omega} [p^2 + (m\omega x)^2 - \underbrace{im\omega[x, p]}_{=i\hbar}]$

$$\boxed{a_- a_+ = \frac{1}{\hbar\omega} H + \frac{1}{2}}$$

Exercise 3

Find: (1) $a_+ a_-$

(2) $[a_-, a_+]$

Solution

The Schrödinger equation can be written as

$$\hbar\omega \left(a_+ a_- + \frac{1}{2} \right) \psi = E \psi$$

or

$$\hbar\omega \left(a_- a_+ - \frac{1}{2} \right) \psi = E \psi.$$

Crucial step: we are now going to prove that if ψ satisfies Schrödinger equation with energy E

$$\hat{H} \psi = E \psi,$$

then $a_+ \psi$ satisfies the Schrödinger equation with energy $(E + \hbar\omega)$:

$$H(a_+ \psi) = (E + \hbar\omega)(a_+ \psi).$$

Proof

$$H(a_+ \psi) = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) (a_+ \psi)$$

$$= \hbar\omega \left(\underbrace{a_+ a_- a_+}_{\text{just moved } a_+ \text{ before } ()} + \frac{1}{2} \underbrace{a_+}_{\text{just moved } a_+ \text{ before } ()} \right) \psi =$$

$$= \hbar\omega a_+ \left(\underbrace{a_- a_+ + \frac{1}{2}}_{= a_+ a_- + 1} \right) \psi$$

$$= a_+ a_- + 1$$

since we proved that
 $[a_-, a_+] = a_- a_+ - a_+ a_- = 1$
 $a_- a_+ = 1 + a_+ a_-$

$$= \hbar\omega a_+ \left(\underbrace{a_+ a_- + \frac{1}{2} + 1}_{\frac{H}{\hbar\omega}} \right) \psi$$

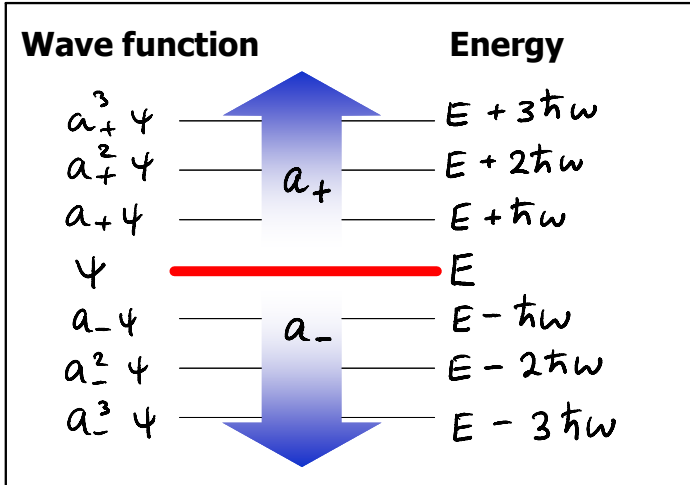
$$= a_+ (H + \hbar\omega) \psi = a_+ (E + \hbar\omega) \psi = (E + \hbar\omega)(a_+ \psi)$$

We can move it here because $(E + \hbar\omega)$ is a constant.

We can also prove that $a_- \psi$ is a solution with energy $(E - \hbar\omega)$:

$$H(a_- \psi) = (E - \hbar\omega)(a_- \psi).$$

Now, if we know one solution, we can generate all solutions using operators a^+ and a^- . This is why these operators are called **ladder operators**. a^+ is a **raising operator** and a^- is a **lowering operator**.



← The "ladder" of states

Problem: what happens if we keep applying lowering operator? Eventually, we get energy that is lower than zero, which we can not have. Therefore, eventually we get the lowest step of the ladder:

$$a_- \psi_0 = 0$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}} (ip + m\omega x)$$

This is our one solution that we need to find all of them!
We will now determine this lowest state:

$$p = -i\hbar \frac{\partial}{\partial x}$$

$$\frac{1}{\sqrt{2\hbar m\omega}} \left(\hbar \frac{d}{dx} + m\omega x \right) \psi_0 = 0$$

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x \psi_0$$

$$\int \frac{1}{\psi_0} d\psi_0 = -\frac{m\omega}{\hbar} \int x dx$$

$$\ln \psi_0 = -\frac{m\omega}{2\hbar} x^2 + \text{constant}$$

$$\psi_0(x) = A e^{-\frac{m\omega}{2\hbar} x^2}$$

Normalization: $|A|^2 \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} dx = |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$

← Lowest state of harmonic oscillator

The corresponding energy is found from

$$\hat{H} \psi_0 = E \psi_0$$

$$\hbar\omega \left(a_+ a_- + \frac{1}{2} \right) \psi_0 = E_0 \psi_0$$

$$\underbrace{\hbar\omega a_+ a_- \psi_0}_{=0} + \frac{\hbar\omega}{2} \psi_0 = E_0 \psi_0 \Rightarrow$$

$$E_0 = \frac{1}{2} \hbar\omega$$

To find all other functions we can use $\psi_n(x) = A_n (a_+)^n \psi_0(x)$

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega$$

Note: we already found all possible energies!

Summary

Quantum harmonic oscillator

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

The ground (lowest) solution of time-independent Schrödinger equation for harmonic oscillator is:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$

$$E_0 = \frac{1}{2} \hbar \omega.$$

To find all other functions we can use $\psi_n(x) = A_n (a_+)^n \psi_0(x)$

The possible energies are: $E_n = \left(n + \frac{1}{2}\right) \hbar \omega$

The ladder operators:

Raising operator $a_+ = \frac{1}{\sqrt{2m\omega\hbar}} (-ip + m\omega x)$

Lowering operator $a_- = \frac{1}{\sqrt{2m\omega\hbar}} (ip + m\omega x)$

Definition of commutator: $[A, B] = AB - BA$

Canonical commutation relation $[x, p] = i\hbar$

Exercise 3

- Find: (1) $a_+ a_-$
(2) $[a_-, a_+]$

Solution

$$\begin{aligned} a_+ a_- &= \frac{1}{2\hbar m\omega} (-ip + m\omega x)(ip + m\omega x) \\ &= \frac{1}{2\hbar m\omega} (p^2 + (m\omega x)^2 - im\omega(px - xp)) \\ &\quad \underbrace{-im\omega(-i\hbar)}_{\substack{'' \\ -m\omega\hbar}} \\ &= \frac{H}{\hbar\omega} - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} [a_-, a_+] &= \frac{1}{2\hbar m\omega} \{ [ip + m\omega x, -ip + m\omega x] \} \\ &= \frac{1}{2\hbar m\omega} \{ [ip, -ip] + [m\omega x, -ip] \\ &\quad + [ip, m\omega x] + [m\omega x, m\omega x] \} = \\ &= \frac{1}{2\hbar m\omega} \{ -m\omega i [x, p] + im\omega [p, x] \} \\ &= \frac{1}{2\hbar m\omega} \{ -m\omega i 2i\hbar \} = 1 \end{aligned}$$