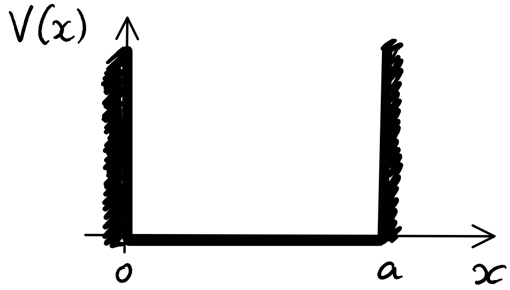
## Lecture 5

### The infinite square well summary:




$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$

$$\boxed{E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}}$$

**Solution:**

Stationary states: $\Psi_n(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}$

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)} \qquad n = 1, 2, \dots$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n}{a} x\right) e^{-i n^2 \pi^2 \hbar t / 2ma^2}$$

Most general solution is a linear combination of stationary states:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \Psi_n(x,t)$$

How to find $c_n$ is $\Psi(x, 0)$ is known?

$t = 0$

Use Fourier's trick:

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\, \Psi(x, 0)\, dx$$

**Example**

A particle in the infinite square well has the initial wave function

$$\Psi(x,0) = Ax(a-x) \qquad (0 \le x \le a)$$

for some constant A. Outside of the well, of course, $\Psi(x,0) = 0$.

Find $\Psi(x,t)$.

**Solution**

We already know that

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-iE_n t/\hbar} \qquad n=1,2,3\ldots$$

We need to find $c_n$ .

$$c_n = \int_0^a \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)}_{\psi_n^*(x)} \psi(x,0)\,dx$$

First, we need to find A (normalize $\Psi(x,0)$ ). $\int_0^a |\Psi(x,0)|^2 dx = 1 \Rightarrow$

$$\int_0^a |A|^2 x^2 (a-x)^2 dx = |A|^2 \frac{a^5}{30} \Rightarrow A = \sqrt{\frac{30}{a^5}}$$

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sqrt{\frac{30}{a^5}}\, x(a-x)\,dx$$

$$= \frac{4\sqrt{15}}{(n\pi)^3}\left[\cos(0) - \cos(n\pi)\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8\sqrt{15}}{(n\pi)^3}, & \text{if } n \text{ is odd} \end{cases}$$

Result:

$$\boxed{\Psi(x,t) = \sqrt{\frac{30}{a}} \left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5\ldots} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-in^2\pi^2\hbar t/2ma^2}}$$

What is $c_n$?

$|c_n|^2$ tells you the probability that a measurement of the energy would yield the value $E_n$. Only the values $E_n$ can be obtained as results of the energy measurements. The sum of all these probabilities will be, of course, 1.

$$\boxed{\sum_{n=1}^{\infty} |c_n|^2 = 1}$$

(see proof in the textbook)

When we write that the wave function $\Psi(x,t)$ has a form

$$\Psi(x,t) = \sum_n c_n \Psi_n(x,t)$$

we say that this state is a superposition of states $\Psi_n(x,t)$.

Let's now explore the concept of superposition further.

**Example**

Suppose the particle starts as a linear combination (superposition) of two stationary states:

$$\Psi(x,0) = c_1 \psi_1(x) + c_2 \psi_2(x)$$

We assume for simplicity that $c_n$ and $\psi_n(x)$ are real. What is the wave function $\Psi(x,t)$ at subsequent times? Find the probability density and describe its motion.

**Solution**

$$\Psi(x,t) = c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2(x) e^{-iE_2 t/\hbar}$$

$$|\Psi(x,t)|^2 = \left(c_1 \psi_1 e^{iE_1 t/\hbar} + c_2 \psi_2 e^{iE_2 t/\hbar}\right) \cdot$$

$$\left(c_1 e^{-iE_1 t/\hbar} + c_2 e^{-iE_2 t/\hbar}\right) = c_1^2 \psi_1^2 + c_2^2 \psi_2^2$$

$$+ c_1 c_2 \psi_1 \psi_2 e^{iE_1 t/\hbar - iE_2 t/\hbar} + c_1 c_2 \psi_1 \psi_2 e^{iE_2 t/\hbar - iE_1 t/\hbar}$$

$$= c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + c_1 c_2 \psi_1 \psi_2 \left(e^{-i\theta} + e^{i\theta}\right)$$

where $\theta = (E_2 - E_1) t/\hbar$

Now we use Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow$

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$\boxed{|\Psi(x,t)|^2 = c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2 c_1 c_2 \psi_1 \psi_2 \cos\left[(E_2 - E_1) t/\hbar\right]}$$

The probability density oscillates sinusoidally, at an angular frequency $(E_2 - E_1)/\hbar$
This is not a stationary state anymore.

**Computer simulation:** http://www.falstad.com/mathphysics.html
1D Quantum mechanics applet