Lecture #26

Review

Postulates of quantum mechanics (1-3)

Postulate 1

The state of a system at any instant of time may be represented by a wave function ψ which is continuous and differentiable. Specifically, if a system is in the state $\psi(\vec{r}, t)$, the average of any physical observable C relevant he this system in time t is

$$\langle C \rangle = \int \psi^* \hat{C} \psi \, d^3r$$

Only normalizable wave functions represent physical states. The set of all square-integrable functions, on a specified interval,

$$f(x) \text{ such that } \int_a^b |f(x)|^2 dx < \infty$$

constitutes a Hilbert space. Wave functions live in Hilbert space.

Postulate 2

To any self-consistently and well-defined observable Q, such as linear momentum, energy, angular momentum, or a number of particles, there correspond an operator $\hat{Q}$ such that measurement of Q yields values (call these measured values q) which are eigenvalues of Q. That is, the values q are those for which the equation

$$\hat{Q}\varphi = q\varphi \leftarrow \text{eigenvalue equation}$$

has a solution φ. The function φ is called the eigenfunction of $\hat{Q}$ corresponding to the eigenvalue q.

Postulate 3

Measurement of the observable Q that yields the value q leaves the system in the state φ_q, where φ_q is the eigenfunction of Q that corresponds to the eigenvalue q.

Generalized statistical interpretation:

If your measure observable Q on a particle in a state $\Psi(x,t)$ you will get one of the eigenvalues of the hermitian operator $\hat{Q}$. If the spectrum of $\hat{Q}$ is discrete, the probability of getting the eigenvalue q_n associated with orthonormalized eigenfunction $f_n(x)$ is

$$|c_n|^2, \quad \text{where} \quad c_n = \langle f_n | \Psi \rangle.$$

It the spectrum is continuous, with real eigenvalues q(z) and associated Dirac-orthonormalized eigenfunctions $f_z(x)$, the probability of getting a result in the range dz is

$$|c(z)|^2 dz, \quad \text{where} \quad c(z) = \langle f_z | \Psi \rangle$$

The wave function "collapses" to the corresponding eigenstate upon measurement.

$$\boxed{\sum_n |c_n|^2 = 1 \quad \text{and} \quad \langle Q \rangle = \sum_n q_n |c_n|^2}$$

Discrete Spectrum

The uncertainty principle:

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

Important differences between classical mechanics and quantum mechanics

1. Superposition $\psi = a\psi_1 + b\psi_2 + c\psi_3 + \ldots$

2. Measurement

3. Entanglement

Let's consider two-level quantum system:

$$\psi = a\psi_1 + b\psi_2 = a|0\rangle + b|1\rangle$$

just designations for ψ_1 and ψ_2

Example:

$|0\rangle$ is spin up $\uparrow$ state $\begin{pmatrix}1\\0\end{pmatrix}$

$|1\rangle$ is spin down $\downarrow$ state $\begin{pmatrix}0\\1\end{pmatrix}$

If we combine two such systems we can build a state

$$\psi_{12} = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right) = \frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)$$

Such state is called maximally "entangled" state since if we measure the first spin being "up", then the measurement of the second spin will yield the result "up" with 100% probability:

$$\psi_{12} = \frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)$$

if we get this result $\Rightarrow$ the wave function ψ_{12} "collapses" to $\psi_{12}' = \uparrow\uparrow \Rightarrow$ the second spin is $\uparrow$ with 100% probability.

If we find the first spin to be $\downarrow$ second spin must be $\downarrow$ as well.

$\psi_{12} \rightarrow \psi_{12}' = \downarrow\downarrow$ after measurement.

Schrödinger equation: summary

The general solution of Schrödinger equation in three dimensions (if V does not depend on time)

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi} \qquad H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

is

$$\psi(\vec{r},t) = \sum_n c_n \psi_n(\vec{r})\, e^{-iE_n t/\hbar},$$

where functions $\psi_n(\vec{r})$ are solutions of time-independent Schrödinger equation

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi}. \qquad [H\psi = E\psi]$$

If potential V is spherically symmetric, i.e. only depends on distance to the origin r, then the separable solutions are

$$\boxed{\psi(r,\theta,\phi) = R(r)\, Y_\ell^m(\theta,\phi)},$$

where $R(r)$ are solutions of radial equation

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left(V(r)-E\right) = \ell(\ell+1)$$

with normalization condition $\int_0^\infty |R|^2 r^2\, dr = 1$

The spherical harmonics are

$$Y_\ell^m(\theta,\phi) = \epsilon\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\; e^{im\phi}\; \underbrace{P_\ell^m(\cos\theta)}_{\text{associated Legendre functions}}$$

$$\epsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m \leq 0 \end{cases} \qquad \boxed{\begin{array}{l} \ell = 0, 1, 2, \ldots \\ m = -\ell, -\ell+1, \ldots 0, 1, \ldots \ell \end{array}}$$

Summary for radial equation:

$$u(r) = rR(r)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

$$\int_0^\infty |u|^2 dr = 1$$

Angular momentum

If
$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y \quad \text{then}$$

Eigenfunctions f_ℓ^m of L^2 and L_z are labeled by m and l:

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m, \qquad L_z f_\ell^m = \hbar m f_\ell^m.$$

$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ (only integer values for orbital angulear momentum)

For a given value of l, there are $2l+1$ values of m: $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$.

Elementary particles carry intrinsic angular momentum **S** in addition to **L**.
Spin of elementary particles has nothing to do with rotation, does not depend on coordinates θ and ϕ, and is purely a quantum mechanical phenomena.

Addition of angular momenta

If you combine any angular momentum j_1 and j_2 you get every value of angular momentum from $|j_1 - j_2|$ to $j_1 + j_2$ in integer steps:

$$j = |j_1 - j_2|, \ldots, (j_1 + j_2)$$

It does not matter if it is orbital angular momentum or spin.

The combined state $|j\,m\rangle$ with total angular momentum j is a linear combination of the composite states:

$$|j m\rangle = \sum_{\substack{m_1, m_2 \\ [m = m_1 + m_2]}} \underbrace{C^{j_1 j_2 j}_{m_1 m_2 m}} \, |j_1 m_1\rangle \, |j_2 m_2\rangle$$

Clebsch-Gordon coefficients.