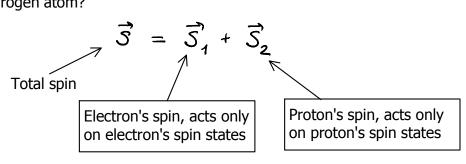
Lecture 25

Addition of angular momenta

Ground state of hydrogen: it has one proton with spin $\sqrt{2}$ and one electron with spin $\sqrt{2}$ (orbital angular momentum is zero). What is the total angular momentum $\overline{3}$ of the hydrogen atom?



The z components just add together and quantum number m for the composite system is simply

 $m = m_1 + m_2$

Combination of two spin $\frac{1}{2}$ particles can carry a total spin of s = 1 or s = 0, depending on whether they occupy the triplet or singlet configuration.

Three states (s_m) with spin s = 1, m = 1, 0, -1:

$$\begin{cases} | 11 \rangle = \uparrow \uparrow \\ | 10 \rangle = \frac{1}{12} (\uparrow \downarrow + \downarrow \uparrow) \\ | 1^{-1} \rangle = \downarrow \downarrow \end{cases}$$

S=1 This is called a **triplet** configuration

and one state with spin s = 0, m = 0:

$$\begin{cases} 100 \\ \forall = \frac{1}{\sqrt{2}} (71 - 17) \\ \end{bmatrix} \quad S = 0 \\ \text{This is called a} \\ \text{singlet configuration.} \end{cases}$$

How do we prove that it is true?

To prove the above, we need to show that

$$5^{2}|SM7 = S(S+1)|SM7$$
, i.e.

for triplet states

$$S^{2}|1M7 = 2|1M7$$
 since $S = 1$ [M=±1,0]

and for singlet states

$$5^2|007 = 0$$
 since $s=0$

This can be shown by direct calculation (see page 186 of the textbook for the proof).

In general, it you combine any angular momentum j_{1} and j_{2} you get every value of angular momentum from $|j_{1} - j_{2}|$ to $j_{1} + j_{2}$ in integer steps:

$$j = j_1 - j_2 j_2, \dots, (j_1 + j_2)$$

It does not matter if it is orbital angular momentum or spin.

Example:

$$j_{1} = \frac{3}{2} \quad j_{2} = 3 = 7$$

$$|j_{1} - j_{2}| = |\frac{3}{2} - 3| = \frac{3}{2}$$

$$j_{1} + j_{2} = \frac{3}{2} + 3 = \frac{9}{2}$$

$$Total angular momentum j can be \qquad \frac{3}{2}, \quad \frac{5}{2}, \quad \frac{1}{2}, \quad an d \quad \frac{9}{2}.$$

Class exercise #11

Quarks carry spin $\frac{1}{2}$. Two quarks (or actually a quark and an antiquark) bind together to make a meson (such as pion or kaon). Three quarks bind together to make a barion (such as proton or neutron). Assume all quarks are in the ground state so the orbital angular momentum is zero).

- (1) What spins are possible for mesons?
- (2) What spins are possible for baryons?

Solution:

(1)
$$S_1 = \frac{1}{2}$$
 $S_2 = \frac{1}{2}$ = $S = 0 \text{ or } 1$.
(2) $S_1 = \frac{1}{2}$ $S_2 = \frac{1}{2}$ $S_3 = \frac{1}{2}$
Add these first
 $S_{12} = 0 \text{ or } 1$

Now add the third spin:

$$S_{12} = 0 \quad S_3 = \frac{1}{2} \quad = 7 \quad S = \frac{1}{2}$$

$$S_{12} = 1 \quad S_3 = \frac{1}{2} \quad = 7 \quad S = \left|\frac{1}{2} - 1\right|_{7.1} \left(\frac{1}{2} + 1\right) = 7$$

$$S = \frac{1}{2} \text{ or } \frac{3}{2}$$

$$5 = \frac{1}{2} \text{ or } \frac{3}{2}$$

L25.P3

The combined state $j \neq 2$ with total angular momentum j is a linear combination of the composite states:

$$\begin{split} ljm7 &= \sum_{m_1,m_2} C_{m_1,m_2m_1}^{j_1j_2j_2} lj_1m_17 lj_2m_27 \\ & [m=m_1+m_2] \\ & Clebsch-Gordon \\ & Coefficients. \end{split}$$

Links to the calculators of Clebsch-Gordan coefficients: http://www.gleet.org.uk/cleb/cgjava.html http://www.volya.net/vc/vc.php

Let's consider our previous case of hydrogen as an example:

$$j_1=\frac{1}{2}$$
 and $j_2=\frac{1}{2}=7$ total j is 1 or 0.

Our states were:

(i)
$$|117 = \uparrow \uparrow$$

 $j_{1}m_{17} = |\frac{1}{2}\frac{1}{2}$
 $|j_{2}m_{2}7 = |\frac{1}{2}\frac{1}{2}7$
 $|117 = |\frac{1}{2}\frac{1}{2}7 |\frac{1}{2}\frac{1}{2}7$

Now we look at the formula above:

$$i_{jm7} = \sum_{m_{1},m_{2}} C_{m_{1}m_{2}m_{2}m_{1}}^{j_{1}j_{2}j_{2}} i_{j_{1}m_{1},m_{2}} i_{j_{2}m_{2}m_{2}}$$

 $m_{1,m_{2}}$
 $[m=m_{1}+m_{2}]$

and compare

$$|117 = |\frac{1}{2}\frac{1}{2}7|\frac{1}{2}\frac{1}{2}7| = 1$$

$$|jm7 = |jm7|\frac{1}{2}m27| = 1$$

$$\frac{\frac{1}{2}\frac{1}{2}1}{\frac{1}{2}\frac{1}{2}1} = 1$$
and all other relevant Clebsch-Gordan coefficients are zero:
$$\frac{d_{11}}{d_{11}}\frac{d_{11}}{d_{11}}\frac{d_{11}}{d_{11}}\frac{d_{12}}{d_{11}}\frac{d_{12}}{d_{12}}\frac{d_{12}}{d$$

(2)
$$1107 = \sqrt{2}(11 + 11)$$

$$1107 = \sqrt{2} |\frac{1}{2} |\frac{1}{2} - \frac{1}{2} \rangle + \sqrt{2} |\frac{1}{2} - \frac{1}{2} \rangle |\frac{1}{2} |\frac{1}{2} \rangle$$
Designations: $\lim_{x \to 0} 7 = \begin{pmatrix} \frac{1}{2} |\frac{1}{2} |\frac{1}{2} - \frac{1}{2} \rangle + \sqrt{2} |\frac{1}{2} |\frac{1}{2} - \frac{1}{2} \rangle |\frac{1}{2} |\frac{1}{2} |\frac{1}{2} \rangle$

$$\int_{1}^{\frac{1}{2} |\frac{1}{2} |$$

All other relevant Clebsch-Gordan coefficients are zero.

$$C_{m_1m_2 0}^{\frac{1}{2} \frac{1}{2} 1} = 0 \quad \text{for all other } m_1 \text{ and } m_2 .$$

L25. P5

Class exercise #12

The electron in a hydrogen atom occupies the combined spin an position state:

$$\Psi = R_{21} \left(\sqrt{1/3} Y_{1}^{\circ} \gamma_{+}^{+} + \sqrt{2/3} Y_{1}^{\dagger} \gamma_{-}^{-} \right)$$

$$\int_{n=2}^{n=2} \ell_{e=1} \int_{m_{e}=0}^{n=1} \frac{1}{m_{s}} \int_{m_{e}=1}^{n=1} \frac{1}{m_{e}} \int_{m_{e}=1}^{n=1} \frac{1}{m_{e$$

Note that $m_{\ell} + m_s = \frac{1}{2}$ in both cases

(a) If you measure the orbital angular momentum squared L^2 , what values might your get and what is the probability of each?

(b) Same for z component of the orbital angular momentum , L_{z} .

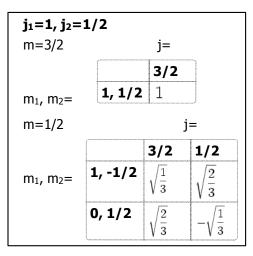
(c) Same for the spin angular momentum squared S^2 .

(d) Same for z component of the spin angular momentum S_{a} .

Let $\vec{J} = \vec{L} + \vec{S}$ be the total angular momentum.

(e) If you measure the total angular momentum squared \mathbf{J}^2 , what values might your get and what is the probability of each?

Clebsch-Gordan coefficients



Hint: this formula is also true

Use it to represent Ψ as a linear superposition of eigenstates $\lim_{n \to \infty} w_n$.

Class exercise #12

The electron in a hydrogen atom occupies the combined spin an position state:

$$\Psi = R_{21} \left(\sqrt{1/3} Y_{1}^{\circ} Y_{1} + \sqrt{2/3} Y_{1}^{\circ} Y_{-} \right)$$

$$\int_{n=2}^{n=2} \ell = 1 \qquad T \qquad T_{S=\frac{1}{2}} \qquad T \qquad T_{S=\frac{1}{2}} \qquad T \qquad T_{S=\frac{1}{2}} \qquad L=1 \qquad M_{s}=-\frac{1}{2} \qquad M_{e}=1 \qquad M_{s}=-\frac{1}{2}$$

Note that $m_{\ell} + m_s = \frac{1}{2}$ in both cases

(a) If you measure the orbital angular momentum squared L^2 , what values might your get and what is the probability of each?

$$l=1$$
 $L^{2} \Psi = l(l+1) t^{2} \Psi = 7$

You get $\hbar^2 \ell(\ell+1) = 2\hbar^2$ with probability P=1 (100%).

(b) Same for z component of the orbital angular momentum, L_2 .

$$L_2 \Psi = \hbar m_e \Psi$$

Possible values of m_{ℓ} : $m_{\ell} = 0$ or 1.

$$P = \left(\left(\frac{1}{3}\right)^2 = \frac{1}{3} \text{ for } m_{\ell} = 0$$

$$P = \left(\left(\frac{1}{3}\right)^2 = \frac{2}{3} \text{ for } m_{\ell} = 1.$$

(c) Same for the spin angular momentum squared S^2 .

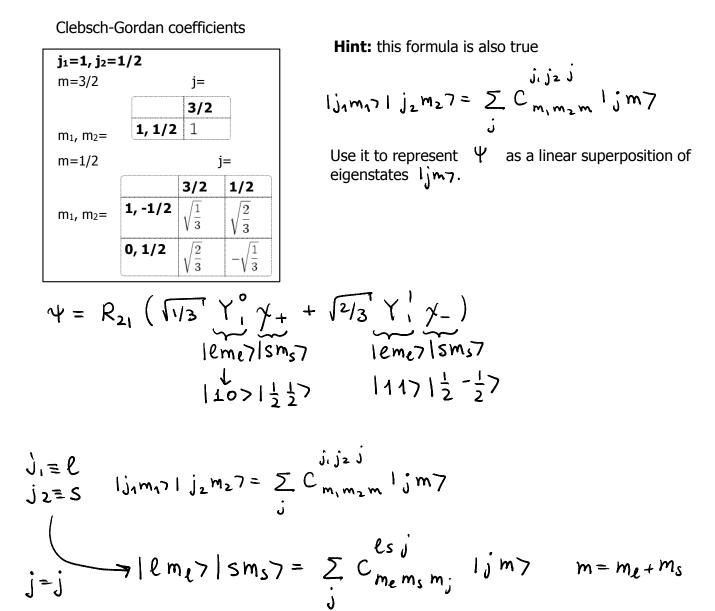
$$S^{2} \Psi = h^{2} s(s+i) \Psi$$
 $s=\frac{1}{2}=7$ you get $\frac{3}{4}h^{2}$ with $P=1$.

(d) Same for z component of the spin angular momentum S_{z} .

$$S_2 = t m_s t m_s = \frac{1}{2}$$
 and $-\frac{1}{2}$
 $P = \frac{1}{3}$ for $m_s = \frac{1}{2}$; $P = \frac{2}{3}$ for $m_s = -\frac{1}{2}$

(e) If you measure the total angular momentum squared \Im^2 , what values might your get and what is the probability of each?

E12



What possible values of j can we have?

$$l=1$$
 $s=\frac{1}{2}$ = $j=l-s$ an $l+s=j=\frac{1}{2}$ and $\frac{3}{2}$

First, we need to express 1071227 vialjm7 :

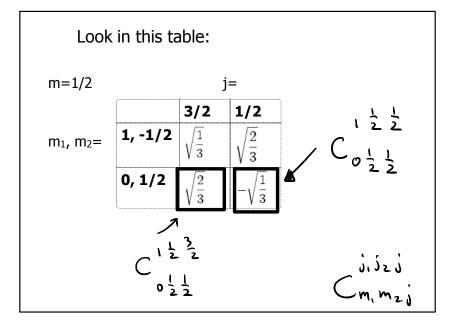
$$\frac{1107}{1227} = C_{0\frac{11}{2}\frac{1}{2}} |\frac{1}{2}\frac{1}{2}7 + C_{0\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}\frac{3}{2}} |\frac{3}{2}\frac{1}{2}7} + C_{0\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}\frac{3}{2}\frac{1}{2}7}$$

$$\frac{1}{12} |\frac{1}{2}\frac{1}{2}7 + C_{0\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}\frac{3}{2}\frac{1}{2}7} + C_{0\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}\frac{3}{2}\frac{1}{2}7}$$

$$\frac{1}{12} |\frac{1}{2}\frac{1}{2}7 + C_{0\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}\frac{3}{2}\frac{1}{2}7} + C_{0\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}7}$$

$$\frac{1}{12} |\frac{1}{2}\frac{1}{2}7 + C_{0\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}7} + C_{0\frac{1}{2}\frac$$

Need: Clebsch-Gordon coefficients:



In the same way,

$$|117|\frac{1}{2}-\frac{1}{2}7 = C \frac{|\frac{1}{2}\frac{1}{2}}{|-\frac{1}{2}\frac{1}{2}} |\frac{1}{2}\frac{1}{2}7 + C \frac{|\frac{1}{2}\frac{3}{2}}{|-\frac{1}{2}\frac{1}{2}} |\frac{3}{2}\frac{1}{2}7$$

(Take coefficients from upper row of the table).

$$= \sqrt{\frac{2}{3}} \frac{1}{2} \frac{1}{2} \frac{1}{2} + \sqrt{\frac{1}{3}} \frac{1}{2} \frac{3}{2} \frac{1}{2}$$

$$\begin{split} \Psi &= R_{21} \left(\sqrt{1/3} Y_{1}^{\circ} \gamma_{+}^{\circ} + \sqrt{2/3} Y_{1}^{\circ} \gamma_{-}^{\circ} \right) \\ &= R_{21} \left(\sqrt{1/3} Y_{1}^{\circ} \gamma_{+}^{\circ} + \sqrt{2/3} Y_{1}^{\circ} \gamma_{-}^{\circ} \right) \\ &= R_{21} \left(\frac{1}{\sqrt{3}} \sqrt{1/3} + \sqrt{2} + \sqrt{2} \sqrt{3} \sqrt{1/3} + \frac{1}{2} - \frac{1}{2} \right) \\ &= R_{21} \left(\frac{1}{\sqrt{3}} \sqrt{1/3} + \sqrt{2} + \sqrt{2} \sqrt{3} + \sqrt{2} - \frac{1}{2} \right) \\ &= R_{21} \left(\frac{1}{\sqrt{3}} \sqrt{1/3} + \sqrt{2} \sqrt{1/3} + \sqrt{2} + \frac{1}{2} \sqrt{2} + \sqrt{2} + \frac{1}{2} \sqrt{2} \right) \\ &+ \sqrt{2} \left(\frac{1}{\sqrt{3}} \sqrt{1/3} + \frac{1}{2} \sqrt{1/3} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \frac{1}{2} \sqrt{2} \right) \\ &+ \sqrt{2} \left(\frac{2\sqrt{2}}{3} - \frac{1}{2} \sqrt{2} + \frac{1}{3} + \frac{1}{2} \sqrt{2} + \sqrt{2} + \frac{1}{3} \sqrt{1/2} + \frac{1}{2} \sqrt{2} \right) \\ &= R_{21} \left(\frac{2\sqrt{2}}{3} - \frac{1}{2} \sqrt{2} + \frac{1}{3} \sqrt{1/2} + \frac{1}{3} \sqrt{1/2} + \frac{1}{2} \sqrt{2} \right) \\ &= R_{21} \left(\frac{2\sqrt{2}}{3} - \frac{1}{2} \sqrt{2} + \frac{1}{3} \sqrt{1/2} + \frac{1}{3} \sqrt{1/2} + \frac{1}{2} \sqrt{2} \right) \\ &= \frac{1}{3} : P = \left(\frac{1}{3} \right)^{2} = \frac{8}{9} \quad \text{for gat} \quad j(j+1) \sqrt{1/2} = \frac{1}{9} \sqrt{1/3} \sqrt{1/3} \\ &= \frac{1}{9} + \sqrt{3} \sqrt{1/3} \sqrt{1/3} + \sqrt{1/3} \sqrt{1/3} \sqrt{1/3} \\ &= \frac{1}{9} + \sqrt{3} \sqrt{1/3} \sqrt{1/3} + \sqrt{1/3} \sqrt{1/3} \sqrt{1/3} + \frac{1}{2} \sqrt{1/3} \sqrt{1/3} + \frac{1}{2} \sqrt{1/3} \right) \\ &= \frac{1}{3} : P = \left(\frac{1}{3} \right)^{2} = \frac{1}{9} + \sqrt{3} \sqrt{1/3} \sqrt{1/3} + \sqrt{1/3} \sqrt{1/3} + \frac{1}{2} \sqrt{1/3} + \frac{1}{2} \sqrt{1/3} \sqrt{1/3} + \frac{1}{3} \sqrt{1/3} \sqrt{1/3} \sqrt{1/3} + \frac{1}{3} \sqrt{1/3} \sqrt{1/3} + \frac{1}{3}$$