Lecture 25

Addition of angular momenta

Ground state of hydrogen: it has one proton with spin $1/2$ and one electron with spin $1/2$ (orbital angular momentum is zero). What is the total angular momentum $\vec{S}$ of the hydrogen atom?

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Total spin

Electron's spin, acts only on electron's spin states

Proton's spin, acts only on proton's spin states

The z components just add together and quantum number m for the composite system is simply

$$\boxed{m = m_1 + m_2}\,.$$

Combination of two spin $1/2$ particles can carry a total spin of s =1 or s = 0, depending on whether they occupy the triplet or singlet configuration.

Three states $|s\ m\rangle$ with spin s = 1, m = 1, 0, -1:

$$\left.\begin{cases} |1\,1\rangle = \uparrow\uparrow \\ |1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,{-1}\rangle = \downarrow\downarrow \end{cases}\right\} \quad s=1$$

This is called a **triplet** configuration.

and one state with spin s = 0, m = 0:

$$\left\{ |0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\} \quad s=0$$

This is called a **singlet** configuration.

How do we prove that it is true?

To prove the above, we need to show that

$$S^2 |SM\rangle = S(S+1)|SM\rangle, \text{ i.e.}$$

for triplet states

$$S^2|1\,M\rangle = 2\,|1\,M\rangle \quad \text{since} \quad S=1 \quad [M=\pm 1, 0]$$

and for singlet states

$$S^2|00\rangle = 0 \quad \text{since} \quad S=0$$

This can be shown by direct calculation (see page 186 of the textbook for the proof).

In general, it you combine any angular momentum j_1 and j_2 you get every value of angular momentum from $|j_1 - j_2|$ to $j_1 + j_2$ in integer steps:

$$j = |j_1 - j_2|, \ldots, (j_1 + j_2)$$

It does not matter if it is orbital angular momentum or spin.

Example:

$$j_1 = \frac{3}{2} \quad j_2 = 3 \quad \Rightarrow$$

$$|j_1 - j_2| = \left|\frac{3}{2} - 3\right| = \frac{3}{2}$$

$$j_1 + j_2 = \frac{3}{2} + 3 = \frac{9}{2}$$

Total angular momentum j can be $\frac{3}{2}, \frac{5}{2}, \frac{7}{2},$ and $\frac{9}{2}$.

Class exercise #11

Quarks carry spin $1/2$. Two quarks (or actually a quark and an antiquark) bind together to make a meson (such as pion or kaon). Three quarks bind together to make a barion (such as proton or neutron). Assume all quarks are in the ground state so the orbital angular momentum is zero).

(1) What spins are possible for mesons?

(2) What spins are possible for baryons?

Solution:

(1) $s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2} \Rightarrow \boxed{s = 0 \text{ or } 1.}$

(2) $\underbrace{s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2}}_{\text{Add these first}} \quad s_3 = \frac{1}{2}$

$s_{12} = 0 \text{ or } 1$

Now add the third spin: $s_{12} = 0 \quad s_3 = \frac{1}{2} \Rightarrow s = \frac{1}{2}$

$s_{12} = 1 \quad s_3 = \frac{1}{2} \Rightarrow s = \left|\frac{1}{2} - 1\right|, \ldots \left(\frac{1}{2} + 1\right) \Rightarrow$

$s = \frac{1}{2} \text{ or } \frac{3}{2}$

$$\boxed{s = \frac{1}{2} \text{ or } \frac{3}{2}}$$

The combined state $|j\,m\rangle$ with total angular momentum j is a linear combination of the composite states:

$$|jm\rangle = \sum_{\substack{m_1, m_2 \\ [m=m_1+m_2]}} \underbrace{C^{j_1 j_2 j}_{m_1 m_2 m}} |j_1 m_1\rangle |j_2 m_2\rangle$$

Clebsch-Gordon coefficients.

Links to the calculators of Clebsch-Gordan coefficients:
http://www.gleet.org.uk/cleb/cgjava.html
http://www.volya.net/vc/vc.php

Let's consider our previous case of hydrogen as an example:

$j_1 = \frac{1}{2}$ and $j_2 = \frac{1}{2} \Rightarrow$ total j is 1 or 0.

Our states were:

(1) $|11\rangle = \uparrow\uparrow$

$|j_1 m_1\rangle = |\frac{1}{2}\frac{1}{2}\rangle$ $\quad$ $|j_2 m_2\rangle = |\frac{1}{2}\frac{1}{2}\rangle$

$$|11\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

Now we look at the formula above:

$$|jm\rangle = \sum_{\substack{m_1, m_2 \\ [m=m_1+m_2]}} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1 m_1\rangle |j_2 m_2\rangle$$

and compare

$$\left.\begin{aligned} |11\rangle &= |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle \\ |jm\rangle &= |j_1 m_1\rangle|j_2 m_2\rangle \end{aligned}\right\} \Rightarrow C^{\frac{1}{2}\frac{1}{2}1}_{\frac{1}{2}\frac{1}{2}1} = 1$$

and all other relevant Clebsch-Gordan coefficients are zero : $C^{\frac{1}{2}\frac{1}{2}1}_{m_1 m_2 1} = 0$, $\quad m_1 \neq 1/2, \; m_2 \neq \frac{1}{2}$

(2) $|10\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

$$|10\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

Designations: $|jm\rangle = C^{\frac{1}{2}\frac{1}{2}1}_{\frac{1}{2}-\frac{1}{2}0}|j_1m_1\rangle|j_2m_2\rangle + C^{\frac{1}{2}\frac{1}{2}1}_{-\frac{1}{2}\frac{1}{2}0}|j_1m_1\rangle|j_2m_2\rangle$

$j=1$
$m=0$

$j_1=\frac{1}{2}$ $m_1=\frac{1}{2}$
$j_2=\frac{1}{2}$ $m_2=-\frac{1}{2}$

$j_1=\frac{1}{2}$ $m_1=-\frac{1}{2}$
$j_2=\frac{1}{2}$ $m_2=\frac{1}{2}$

$$C^{\frac{1}{2}\frac{1}{2}1}_{\frac{1}{2}-\frac{1}{2}0} = \frac{1}{\sqrt{2}}$$

$$C^{\frac{1}{2}\frac{1}{2}1}_{-\frac{1}{2}\frac{1}{2}0} = \frac{1}{\sqrt{2}}$$

All other relevant Clebsch-Gordan coefficients are zero.

$C^{\frac{1}{2}\frac{1}{2}1}_{m_1m_20} = 0$ for all other m_1 and m_2 .

Class exercise #12

The electron in a hydrogen atom occupies the combined spin an position state:

$$\psi = \underset{n=2,\ \ell=1}{R_{21}} \left(\sqrt{1/3}\ \underbrace{Y_1^0}_{\substack{\ell=1 \\ m_\ell=0}}\ \underbrace{\chi_+}_{\substack{s=\frac{1}{2} \\ m_s=\frac{1}{2}}} + \sqrt{2/3}\ \underbrace{Y_1^1}_{\substack{\ell=1 \\ m_\ell=1}}\ \underbrace{\chi_-}_{\substack{s=\frac{1}{2} \\ m_s=-\frac{1}{2}}} \right)$$

Note that $m_\ell + m_s = \frac{1}{2}$ in both cases

(a) If you measure the orbital angular momentum squared L^2, what values might your get and what is the probability of each?

(b) Same for z component of the orbital angular momentum , L_z.

(c) Same for the spin angular momentum squared S^2.

(d) Same for z component of the spin angular momentum S_z .

Let $\vec{J} = \vec{L} + \vec{S}$ be the total angular momentum.

(e) If you measure the total angular momentum squared J^2, what values might your get and what is the probability of each?

Clebsch-Gordan coefficients

j_1=1, j_2=1/2

m=3/2

m_1, m_2= \ j=	**3/2**
1, 1/2	1

m=1/2

m_1, m_2= \ j=	**3/2**	**1/2**
1, -1/2	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
0, 1/2	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$

Hint: this formula is also true

$$|j_1 m_1\rangle\, |j_2 m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m}\, |j\, m\rangle$$

Use it to represent ψ as a linear superposition of eigenstates $|j m\rangle$.

Class exercise #12

The electron in a hydrogen atom occupies the combined spin an position state:

$$\psi = \underset{\substack{\uparrow \\ n=2}}{R}_{2\underset{\ell=1}{1}} \left(\sqrt{1/3}\, \underbrace{Y_1^0}_{\substack{\ell=1 \\ m_\ell=0}} \underbrace{\chi_+}_{\substack{s=\frac{1}{2} \\ m_s=\frac{1}{2}}} + \sqrt{2/3}\, \underbrace{Y_1^1}_{\substack{\ell=1 \\ m_\ell=1}} \underbrace{\chi_-}_{\substack{s=\frac{1}{2} \\ m_s=-\frac{1}{2}}} \right)$$

Note that $m_\ell + m_s = \frac{1}{2}$ in both cases

(a) If you measure the orbital angular momentum squared L^2, what values might your get and what is the probability of each?

$\ell = 1 \qquad L^2 \psi = \ell(\ell+1)\hbar^2 \psi \quad \Rightarrow$

You get $\hbar^2 \ell(\ell+1) = 2\hbar^2$ with probability P=1 (100%).

(b) Same for z component of the orbital angular momentum, L_z.

$L_z \psi = \hbar m_\ell \psi$

Possible values of m_ℓ: $m_\ell = 0$ or 1.

$P = \left(\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{3}$ for $m_\ell = 0$

$P = \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$ for $m_\ell = 1$.

(c) Same for the spin angular momentum squared S^2.

$S^2 \psi = \hbar^2 s(s+1) \psi \qquad s = \frac{1}{2} \Rightarrow$ You get $\frac{3}{4}\hbar^2$ with $P=1$.

(d) Same for z component of the spin angular momentum S_z.

$S_z \psi = \hbar m_s \psi \qquad m_s = \frac{1}{2}$ and $-\frac{1}{2}$

$P = 1/3$ for $m_s = \frac{1}{2}$; $P = 2/3$ for $m_s = -\frac{1}{2}$

E12

(e) If you measure the total angular momentum squared J^2, what values might your get and what is the probability of each?

Clebsch-Gordan coefficients

$j_1=1$, $j_2=1/2$

m=3/2 j=

m1, m2=

	3/2
1, 1/2	1

m=1/2 j=

m1, m2=

	3/2	1/2
1, -1/2	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
0, 1/2	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$

Hint: this formula is also true

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m} |j\,m\rangle$$

Use it to represent Ψ as a linear superposition of eigenstates $|jm\rangle$.

$$\Psi = R_{21}\left(\sqrt{1/3}\,\underbrace{Y_1^0}_{|\ell m_\ell\rangle}\underbrace{\chi_+}_{|s m_s\rangle} + \sqrt{2/3}\,\underbrace{Y_1^1}_{|\ell m_\ell\rangle}\underbrace{\chi_-}_{|s m_s\rangle}\right)$$

$$|1 0\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle \qquad |1 1\rangle|\tfrac{1}{2} -\tfrac{1}{2}\rangle$$

$j_1 \equiv \ell$

$j_2 \equiv s$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_j C^{j_1 j_2 j}_{m_1 m_2 m} |j\,m\rangle$$

$j=j$

$$\rightarrow |\ell m_\ell\rangle |s m_s\rangle = \sum_j C^{\ell s j}_{m_\ell m_s m_j} |j\,m\rangle \qquad m = m_\ell + m_s$$

What possible values of j can we have?

$\ell = 1 \quad s = \frac{1}{2} \Rightarrow j = \ell - s$ an $\ell + s \Rightarrow j = \frac{1}{2}$ and $\frac{3}{2}$

First, we need to express $|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$ via $|jm\rangle$:

$l \quad s \quad j \qquad |jm\rangle$

$$\underbrace{|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle}_{\boxed{|l m_l\rangle|s m_s\rangle}} = C^{1\,\frac{1}{2}\,\frac{1}{2}}_{0\,\frac{1}{2}\,\frac{1}{2}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle + C^{1\,\frac{1}{2}\,\frac{3}{2}}_{0\,\frac{1}{2}\,\frac{1}{2}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle$$

$m_l \quad m_s \quad m = m_l + m_s$

Need: Clebsch-Gordon coefficients:

Look in this table:

$m=1/2$ $\qquad$ $j=$

$m_1, m_2=$

	3/2	**1/2**
1, -1/2	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
0, 1/2	$\boxed{\sqrt{\frac{2}{3}}}$	$\boxed{-\sqrt{\frac{1}{3}}}$

$C^{1\,\frac{1}{2}\,\frac{1}{2}}_{0\,\frac{1}{2}\,\frac{1}{2}}$

$C^{1\,\frac{1}{2}\,\frac{3}{2}}_{0\,\frac{1}{2}\,\frac{1}{2}}$

$C^{j_1 j_2 j}_{m_1 m_2 j}$

$$|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = -\frac{1}{\sqrt{3}}|\tfrac{1}{2}\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle$$

In the same way,

$$|11\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = C^{1\,\frac{1}{2}\,\frac{1}{2}}_{1\,-\frac{1}{2}\,\frac{1}{2}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle + C^{1\,\frac{1}{2}\,\frac{3}{2}}_{1\,-\frac{1}{2}\,\frac{1}{2}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle$$

(Take coefficients from upper row of the table).

$$= \sqrt{\frac{2}{3}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle$$

$$\psi = R_{21}\left(\sqrt{1/3}\,\underbrace{Y_1^0}_{|\ell m_\ell\rangle}\,\underbrace{\chi_+}_{|s m_s\rangle} + \sqrt{2/3}\,\underbrace{Y_1^1}_{|\ell m_\ell\rangle}\,\underbrace{\chi_-}_{|s m_s\rangle}\right)$$

$$|1\,0\rangle\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \qquad |1\,1\rangle\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

$$= R_{21}\left(\frac{1}{\sqrt{3}}\,|1\,0\rangle\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1\,1\rangle\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right)$$

plug in from previous page

$$\psi = R_{21}\left\{\frac{1}{\sqrt{3}}\left(\sqrt{\frac{2}{3}}\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right) + \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{3}}\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right)\right\}$$

$$= R_{21}\left(\frac{2\sqrt{2}}{3}\underbrace{\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle}_{|j m\rangle} + \frac{1}{3}\underbrace{\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle}_{|j m\rangle}\right)$$

$j = \frac{3}{2}$: $P = \left|\frac{2\sqrt{2}}{3}\right|^2 = \frac{8}{9}$ to get $j(j+1)\hbar^2 = \frac{15}{4}\hbar^2$

$j = \frac{1}{2}$: $P = \left|\frac{1}{3}\right|^2 = \frac{1}{9}$ to get $j(j+1)\hbar^2 = \frac{3}{4}\hbar^2$

since $J^2|j m\rangle = j(j+1)\hbar^2|j m\rangle$