Lecture 24

Stern-Gerlach experiment, computer simulation.

http://phet.colorado.edu/new/simulations/sims.php?sim=SternGerlach_ExperimentDuring

Results of the previous lecture:

S_z: eigenfunctions $\begin{cases} \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$ eigenvalues $\begin{cases} +\hbar/2 & (\text{spin up}\uparrow) \\ -\hbar/2 & (\text{spin down}\downarrow) \end{cases}$

If your measure S_z for the general spin state $\chi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

your get $\hbar/2$ with probability $|a|^2$, and $-\hbar/2$ with probability $|b|^2$.

S_x: eigenfunctions $\begin{cases} \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$ eigenvalues $\begin{cases} +\hbar/2 \\ -\hbar/2 \end{cases}$

The general state above $\chi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

can be written in terms of eigenfunctions of S_x as

$$\chi = \frac{a+b}{\sqrt{2}}\chi_+^{(x)} + \frac{a-b}{\sqrt{2}}\chi_-^{(x)} .$$

Therefore, if you measure S_x on this state, you get $\frac{\hbar}{2}$ with probability $\frac{|a+b|^2}{2}$ and $-\frac{\hbar}{2}$ with probability $\frac{|a-b|^2}{2}$.

Remember that after measurement the wave function "collapses" to the eigenfunction that corresponds to the eigenvalue that your got.

Simulation 1

1 magnet, initial state $+z$

What results do we get for S_z measurement?

Here, eigenvalue $\hbar/2$ is "up" and eigenvalue $-\hbar/2$ corresponds to "down".

Our initial state: $\chi = \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ [for $+z$] $\Rightarrow$ $\begin{matrix} a=1 \\ b=0 \end{matrix}$

and we get "up" with 100% probability. Our wave function then stays to be χ_+ since it is already in the eigenstate corresponding to result "up".

Simulation 2

1 magnet, initial state +x

What result do we get for S_z measurement?

Our initial state: $\chi = \chi_+^{(x)}$ [for $+x$] $\Rightarrow$

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\chi_+} + \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\chi_-}$$

χ_+, χ_- $\rightarrow$ eigenstates of S_z

$$\chi = \underbrace{\frac{1}{\sqrt{2}}}_{a}\chi_+ + \underbrace{\frac{1}{\sqrt{2}}}_{b}\chi_- = a\chi_+ + b\chi_-$$

$|a|^2 = 1/2$, $|b|^2 = 1/2$

Therefore, we get result "up" with 50% probability and result "down" with 50% probability. Our spin state after measurement is χ_+ if we got result "up" and χ_- if we got result "down".

Note: we get the same results if our initial spin state was "random xz".

Simulation 3

3 magnets with orientations z, x, and z.

Initial spin state: random xz.

Magnet 1 only lets +z component through, magnet 2 only lets +x component through.

Question: what results do we get after the third magnet?
Do we just get + z since magnet 1 does not let -z through or do we still get both +z and -z? If so, with what probabilities?

(1) Our state after the first magnet is $\chi = \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since

we only allow "up" states through. Remember, if we made a measurement of S_z and got result "up" (+z), then the wave function collapses to the corresponding eigenfunction χ_+.

(2) From our previous lecture, we can write any state $\chi = a\chi_+ + b\chi_-$

as $\chi = \frac{a+b}{\sqrt{2}}\chi_+^{(x)} + \frac{a-b}{\sqrt{2}}\chi_-^{(x)}$.

We have $\chi = \chi_+ \Rightarrow a = 1$ and $b = 0 \Rightarrow$

$$\chi = \frac{1}{\sqrt{2}}\chi_+^{(x)} + \frac{1}{\sqrt{2}}\chi_-^{(x)}.$$

We can also prove it just by algebraic manipulations

$$\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+1 \\ 1-1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\chi_+^{(x)}} + \frac{1}{\sqrt{2}}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\chi_-^{(x)}} = \frac{1}{\sqrt{2}}\chi_+^{(x)} + \frac{1}{\sqrt{2}}\chi_-^{(x)} \quad \text{QED.}$$

If we make a measurement of S_x on our state

$$\chi = \underset{\alpha}{\underset{\uparrow}{\frac{1}{\sqrt{2}}}}\ \underset{\text{"up"}}{\chi_+^{(x)}} + \underset{\beta}{\underset{\uparrow}{\frac{1}{\sqrt{2}}}}\ \underset{\text{"down"}}{\chi_-^{(x)}}$$

we get result "up" with probability 50% [$|\alpha|^2 = \frac{1}{2}$] and result "down" with probability 50% [$|\beta|^2 = \frac{1}{2}$].

Now, magnet 2 only lets "up" spins through, so our wave function after magnet 2 is

$$\boxed{\chi = \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}$$

(3) Now we can answer our question: what happens if we measure S_z again with our third magnet?

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\underset{\substack{\uparrow\\ \chi_+ \\ \text{for } S_z}}{\begin{pmatrix}1\\0\end{pmatrix}} + \frac{1}{\sqrt{2}}\underset{\substack{\uparrow\\ \chi_- \\ \text{for } S_z}}{\begin{pmatrix}0\\1\end{pmatrix}} = \underset{\substack{\uparrow\\ a}}{\frac{1}{\sqrt{2}}}\chi_+ + \underset{\substack{\uparrow\\ b}}{\frac{1}{\sqrt{2}}}\chi_-$$

$|a|^2 = \frac{1}{2}$ "up"

$|b|^2 = 1/2$ "down"

Therefore, we get results "up" (+z) and "down" (-z) with equal 50% probabilities !

THE END!

Addition of angular momenta

Let's go back to ground state of hydrogen: it has one proton with spin $1/2$ and one electron with spin $1/2$ (orbital angular momentum is zero). What is the total angular momentum $\vec{S}$ of the hydrogen atom?

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Total spin

Electron's spin, acts only on electron's spin states

Proton's spin, acts only on proton's spin states

$$S_z \chi_1 \chi_2 = \left(S_z^{(1)} + S_z^{(2)} \right) \chi_1 \chi_2 =$$

electron's spin state

proton's spin state

$$= \left(S_z^{(1)} \chi_1 \right) \chi_2 + \chi_1 \left(S_z^{(2)} \chi_2 \right) = \hbar m_1 \chi_1 \chi_2 + \hbar m_2 \chi_1 \chi_2$$

$$= \hbar \left(m_1 + m_2 \right) \chi_1 \chi_2$$

Therefore, the z components just add together and quantum number m for the composite system is simply

$$\boxed{m = m_1 + m_2}\,.$$

There are four possible combinations:

$m_1 = \frac{1}{2}$ $m_2 = \frac{1}{2}$ $\uparrow\uparrow$ $m = 1$

$m_1 = \frac{1}{2}$ $m_2 = -\frac{1}{2}$ $\uparrow\downarrow$ $m = 0$

$m_1 = -\frac{1}{2}$ $m_2 = -\frac{1}{2}$ $\downarrow\uparrow$ $m = 0$

$m_1 = -\frac{1}{2}$ $m_2 = -\frac{1}{2}$ $\downarrow\downarrow$ $m = -1$

(first arrow corresponds to the electron spin and second arrow corresponds to the nuclear spin) .

Well, it appears that we have an extra state!

Let's apply lowering operator to state $\uparrow\uparrow$ to sort this out :

$$S_- |s\, m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\, |s\;\, m-1\rangle$$

$$S_-^{(1)} \uparrow = S_-^{(1)} \left|\tfrac{1}{2}\;\tfrac{1}{2}\right\rangle = \hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}\left|\tfrac{1}{2}\; -\tfrac{1}{2}\right\rangle = \hbar\downarrow$$

state with $s = 1/2,\ m = 1/2$

state with $s = \frac{1}{2}\ \ m = -\frac{1}{2}$

$$S_-(\uparrow\uparrow) = \left(S_-^{(1)} + S_-^{(2)}\right)\uparrow\uparrow$$

$$= \left(S_-^{(1)}\uparrow\right)\uparrow + \uparrow\left(S_-^{(2)}\uparrow\right) = \hbar\downarrow\uparrow + \hbar\uparrow\downarrow = \hbar\left(\downarrow\uparrow + \uparrow\downarrow\right)$$

(Note: normalization is not preserved here).

So we can sort out four states as follows:

Three states $|s\; m\rangle$ with spin s = 1, m = 1, 0, -1 :

$$\left.\begin{cases} |1\,1\rangle = \uparrow\uparrow \\ |1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,{-1}\rangle = \downarrow\downarrow \end{cases}\right\}\ S=1$$

This is called a **triplet** configuration.

and one state with spin s = 0, m = 0:

$$\left\{ |0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\}\quad S=0$$

This is called a **singlet** configuration.

Class exercise:

(1) Apply lowering operator S_- to state $|10\rangle$, what do you get?

$$S_-^{(1)} \uparrow = \hbar \downarrow \quad \text{(from previous page)}$$

$$S_+^{(1)} \uparrow = 0$$

$$S_-|10\rangle = \left(S_-^{(1)} + S_-^{(2)}\right) \frac{1}{\sqrt{2}} \left(\uparrow\downarrow + \downarrow\uparrow \right) = \frac{1}{\sqrt{2}} \left(S_-^{(1)} \uparrow \right) \downarrow$$

$$+ \frac{1}{\sqrt{2}} \uparrow \left(\cancel{S_-^{(2)} \downarrow} \right)^{\to 0} + \frac{1}{\sqrt{2}} \left(\cancel{S_-^{(1)} \downarrow} \right)^{\to 0} \uparrow + \frac{1}{\sqrt{2}} \downarrow \left(S_-^{(2)} \uparrow \right)$$

$$= \frac{1}{\sqrt{2}} \hbar \downarrow\downarrow + \frac{1}{\sqrt{2}} \hbar \downarrow\downarrow = \frac{1}{\sqrt{2}} \hbar \cdot 2 \downarrow\downarrow = \sqrt{2} \hbar \downarrow\downarrow = \sqrt{2} \hbar \, |1 \; -1\rangle$$

(2) Apply raising and lowering operators $S_\pm$ to $|00\rangle$ state, what do you get?

$$S_-|00\rangle = S_- \frac{1}{\sqrt{2}} \left(\uparrow\downarrow - \downarrow\uparrow \right) =$$

$$= \frac{1}{\sqrt{2}} \left(S_-^{(1)} \uparrow \right) \downarrow + \frac{1}{\sqrt{2}} \uparrow \left(\cancel{S_-^{(2)} \downarrow} \right)^{\to 0}$$

$$- \frac{1}{\sqrt{2}} \left(\cancel{S_-^{(1)} \downarrow} \right)^{\to 0} \uparrow - \frac{1}{\sqrt{2}} \downarrow \left(S_-^{(2)} \uparrow \right) = \frac{1}{\sqrt{2}} \hbar \downarrow\downarrow - \frac{1}{\sqrt{2}} \hbar \downarrow\downarrow = \underline{\underline{0}}$$

$$S_+^{(1)} \downarrow = S_+^{(1)} \left| \tfrac{1}{2} \; -\tfrac{1}{2} \right\rangle = \underbrace{\sqrt{\tfrac{1}{2} \cdot \tfrac{3}{2} - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2} + 1\right)}}_{=1} \; \hbar \left| \tfrac{1}{2} \; \tfrac{1}{2} \right\rangle = \hbar \uparrow$$

$$S_+|00\rangle = \left(S_+^{(1)} + S_+^{(2)} \right) \left(\uparrow\downarrow - \downarrow\uparrow \right)$$

$$= \frac{1}{\sqrt{2}} \left(\cancel{S_+^{(1)} \uparrow} \right)^{\to 0} \downarrow + \frac{1}{\sqrt{2}} \uparrow \left(S_+^{(2)} \downarrow \right) - \frac{1}{\sqrt{2}} \left(S_+^{(1)} \downarrow \right) \uparrow - \frac{1}{\sqrt{2}} \downarrow \left(\cancel{S_+^{(2)} \uparrow} \right)^{\to 0}$$

$$= \frac{1}{\sqrt{2}} \hbar \uparrow\uparrow - \frac{1}{\sqrt{2}} \hbar \uparrow\uparrow = \underline{\underline{0}}$$

Summary: Combination of two spin $1/2$ particles can carry a total spin of s =1 or s = 0, depending on whether they occupy the triplet or singlet configuration.