**Lecture 23**

**Spin, problem solving.**

During the previous lecture, we derived expressions for the three spin matrixes:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We write for convenience

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma},$$

where

$$\boxed{\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

are called Pauli spin matrices.

**<u>Question 1</u>**

If you measure $S_z$ on a particle in a general state

$$\chi = a\chi_+ + b\chi_- = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

what result will your get?

**Answer:**
$\chi_-$ and $\chi_+$ are eigenfunctions of $S_z$ with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. Therefore, you get $\frac{\hbar}{2}$ with probability $|a|^2$ or $-\frac{\hbar}{2}$ with probability $|b|^2$.

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**Class exercise 9**

## Question 2

If you measure $S_x$ on a particle in a general state

$$\chi = a\chi_+ + b\chi_- = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

what result will your get?

Hint: need to find eigenvalues and eigenfunctions or $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

## Solution

To find eigenvalues, we solve characteristic equation

$$\det(S_x - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{\hbar^2}{4} = 0 \quad \Rightarrow \quad \boxed{\lambda = \pm\frac{\hbar}{2}.}$$

Next, we find the corresponding eigenfunctions $\chi_+^{(x)}, \chi_-^{(x)}$

(1) $\lambda_1 = \frac{\hbar}{2}$ $\qquad S_x \chi_+^{(x)} = \frac{\hbar}{2}\chi_+^{(x)}$

$$S_x\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$\begin{pmatrix}\beta\\\alpha\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \qquad \alpha = \beta$$

Normalization $\quad (\alpha \;\; \alpha)\begin{pmatrix}\alpha \\ \alpha\end{pmatrix} = 1 \;\Rightarrow\; 2|\alpha|^2 = 1$

$$\alpha = \frac{1}{\sqrt{2}} \qquad \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix}$$

(2) $\quad \lambda = -\frac{\hbar}{2}$

$$\boxed{S_x \chi_-^{(x)} = -\frac{\hbar}{2}\chi_-^{(x)}}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \;\Rightarrow\; \alpha = -\beta$$

$$(\alpha \;\; -\alpha)\begin{pmatrix}\alpha \\ -\alpha\end{pmatrix} = 2|\alpha|^2 = 1 \;\Rightarrow\; \alpha = \frac{1}{\sqrt{2}}$$

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix}$$

Summary: the normalized eigenfunctions of $S_x$ are $\chi_\pm^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ \pm 1\end{pmatrix}$

with corresponding eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ .

Vector $\quad \chi = a\chi_+ + b\chi_-$

can be expressed as a linear combination of these eigenfunctions:

$$\chi = c\,\chi_+^{(x)} + d\,\chi_-^{(x)}$$

To find the expansion coefficients c and d, we as before need to find the

$$c = \langle \chi_+^{(x)} | \chi \rangle \qquad d = \langle \chi_-^{(x)} | \chi \rangle$$

$$c = \langle \chi_+^{(x)} | \chi \rangle$$
$$d = \langle \chi_-^{(x)} | \chi \rangle$$

(1) $c = \langle \chi_+^{(x)} | \chi \rangle = \langle \chi_+^{(x)} | (a\chi_+ + b\chi_-) \rangle$

$$= \frac{1}{\sqrt{2}} a \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} b \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (a+b)$$

(2) $d = \langle \chi_-^{(x)} | \chi \rangle = \langle \chi_-^{(x)} | (a\chi_+ + b\chi_-) \rangle$

$$= \frac{1}{\sqrt{2}} a \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} b \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (a-b)$$

Then, $$\boxed{\chi = \frac{(a+b)}{\sqrt{2}} \chi_+^{(x)} + \frac{(a-b)}{\sqrt{2}} \chi_-^{(x)}}$$

**Answer:**

If you measure $S_x$ you get $\frac{\hbar}{2}$ with probability $\frac{|a+b|^2}{2}$ and $-\frac{\hbar}{2}$ with probability $\frac{|a-b|^2}{2}$.

**Class exercise 10**

Suppose a spin-$\frac{1}{2}$ particle is in the state $\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

What are the probabilities of getting $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ if you measure $S_z$ and $S_x$ ?

(1) $S_z$ $\quad \chi = a\chi_+ + b\chi_- = \frac{1}{\sqrt{6}}(1+i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{6}}\, 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$

$$a = \frac{1+i}{\sqrt{6}} \quad \text{and} \quad b = \frac{2}{\sqrt{6}}.$$

Therefore, the corresponding probabilities are

$$|a|^2 = \frac{(1-i)(1+i)}{6} = \frac{1}{3} \quad \text{for } \frac{\hbar}{2}$$

$$|b|^2 = \frac{4}{6} = \frac{2}{3} \quad \text{for } -\frac{\hbar}{2}.$$

Check $|a|^2 + |b|^2 = \frac{1}{3} + \frac{2}{3} = 1$ ok

(2) $S_x$ $\quad \frac{|a+b|^2}{2} = \frac{(1-i+2)(3+i)}{2\cdot 6} = \frac{5}{6}$

probability of $\hbar/2$ result

$$\frac{|a-b|^2}{2} = \frac{(-1-i)(-1+i)}{2\cdot 6} = \frac{1}{6}$$

probability of $-\hbar/2$ result.