Lecture 2

Summary:

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi}$$

$$P_{ab} = \int_a^b \rho(x)\, dx$$

$$1 = \int_{-\infty}^{\infty} \rho(x)\, dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, \rho(x)\, dx$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\, \rho(x)\, dx$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

From our previous results

$$\boxed{\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1} \qquad \text{Eq. (2.1)}$$

If $\Psi(x,t)$ is a solution $\Rightarrow$ $A\Psi(x,t)$ is also a solution $\Rightarrow$

We need to find such A that Eq. (2.1) is satisfied

(i.e. normalize the wave function $\Psi(x,t)$).

Physically realizable states correspond to the square-integrable solutions of the Schrödinger equation. If we normalize the wave function at time t=0, it will stay normalized. **Schrödinger equation automatically preserves the normalization of the wave function as we will prove below.**

We need to prove that $\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 0$

Then, if Ψ is normalized at $t=0 \Rightarrow$ it stays normalized for all future times.
Below, we will omit $\pm\infty$.

Proof

$$\frac{d}{dt}\int |\Psi(x,t)|^2 dx = \int \frac{\partial}{\partial t} |\Psi(x,t)|^2 dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^* \Psi) = \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

Using shrödinger equation $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi$$

Complex conjugate: $\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$

$\begin{pmatrix} i \to -i \\ \Psi \to \Psi^* \end{pmatrix}$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right) =$$

$$= \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right]$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty} = 0$$

$\Psi(x,t) \to 0$ as $x \to \pm\infty$
(otherwise, wave function will not be normalizable)

Exercise #1

Problem 1.5

Consider the wave function $\Psi(x,t) = A e^{-\lambda|x|} e^{-i\omega t}$,

where A, λ, and ω are positive real constants.

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x and mark the points ($\langle x\rangle+\sigma$) and ($\langle x\rangle-\sigma$), to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside of this range?

Solution

> **How to find complex conjugate:**
>
> (1) Switch $i \to -i$
> (2) For all complex variables or constants change $a \to a^*$.

$$\Psi = A e^{-\lambda|x|} e^{-i\omega t}$$

$$\Psi^* = A e^{-\lambda|x|} e^{i\omega t}$$

since A, λ, and ω are positive real constants.

(a) $$1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \Psi^* \Psi \, dx =$$

$$= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} \cancelto{=1}{e^{-i\omega t} e^{i\omega t}} dx = A^2 \int_{-\infty} e^{-2\lambda|x|} dx$$

$$= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx = 2A^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Bigg|_0^{\infty} = \frac{A^2}{\lambda}$$

> Note : $$\int_{x_1}^{x_2} e^{ax} dx = \frac{e^{ax}}{a}\Bigg|_{x_1}^{x_2}$$

Exercise 1

$$1 = \frac{A^2}{\lambda} \Rightarrow \boxed{A = \sqrt{\lambda}}$$

(b) Calculate $\langle x \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = A^2 \int_{-\infty}^{\infty} x\, e^{-2\lambda |x|} dx =$$

$$= \lambda \left[\underbrace{\int_{-\infty}^{0} x\, e^{-2\lambda |x|} dx}_{= -\int_0^\infty e^{-2\lambda x} dx} + \int_{0}^{\infty} x\, e^{-2\lambda x} dx \right] = 0$$

Note: In this case, integrand $x e^{-2\lambda |x|}$ is odd, therefore $\int_{-\infty}^{\infty}$ is zero as shown above since $\int_{-\infty}^{0} x e^{-2\lambda |x|} dx = -\int_0^\infty x\, e^{-2\lambda x} dx$.

Calculate $\langle x^2 \rangle$:

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^2 e^{-2\lambda x} dx = 2A^2 \int_0^\infty x^2 e^{-2\lambda |x|} dx$$

even integrand $\Rightarrow$ $\int_{-\infty}^{0} f(x) dx = \int_0^\infty f(x) dx$

$$= 2\lambda\, 2 \frac{1}{(2\lambda)^3} = \frac{1}{2\lambda^2}$$

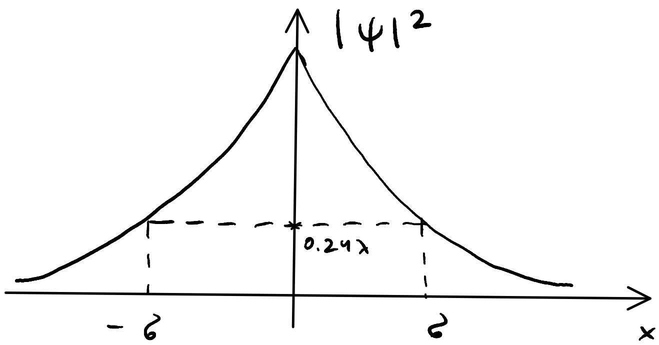
We used $\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1}$

$$\left.\boxed{\begin{array}{l} \langle x \rangle = 0 \\ \langle x^2 \rangle = \frac{1}{2\lambda^2} \end{array}}\right\} \Rightarrow \sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2}\lambda}$$

Exercise 1

$$|\psi(\pm\sigma)|^2 = A^2 e^{-2\lambda\sigma} = \lambda\, e^{-2\cancel{\lambda}/\sqrt{2}\cancel{\lambda}}$$

$$= \lambda e^{-\sqrt{2}} = 0.2431\,\lambda$$



Probability to find the particle outside of region corresponding to ± one standard deviation:

$$P = 2\int_{\sigma}^{\infty} |\psi|^2\, dx = 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x}\, dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Bigg|_{\sigma}^{\infty}$$

$$= -\cancel{2}\lambda\, \frac{e^{-2\lambda\sigma}}{-2\cancel{\lambda}} = e^{-2\lambda\frac{1}{\sqrt{2}\,\lambda}} = e^{-\sqrt{2}} = 0.2431$$