Lecture #16

Example 1:

Imagine a system in which there are just two linearly independent states

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The most general state is a normalized linear combination

$$|S\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

The Hamiltonian can be expressed as a hermitian matrix, suppose it has a form

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix},$$

where h and g are real constants. If the system starts out (at t=0) in state $|1\rangle$ what is its state in time t?

Solution

The time-dependent Schrödinger equation is $i\hbar \frac{d}{dt}|S\rangle = H|S\rangle.$

The general solution of this equation is

$$|S\rangle = \sum_{n=1,2} C_n |S\rangle_n e^{-iE_n t/\hbar}$$

where $|S\rangle_n$ are solutions of time-independent Schrödinger equation

$$H|S\rangle_n = E_n |S\rangle_n$$

Therefore, we need to find eigenvalues and eigenfunctions of the Hamiltonian H, i.e. find eigenvalues and eigenfunctions of 2x2 matrix

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}.$$

We will also later need to find C_n.

To find the eigenvalues E of a matrix we solve characteristic equation

$$\det(H - E\underset{\uparrow}{I}) = 0$$

$\nwarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix

$$\det(H - EI) = \det\left[\begin{pmatrix} h & g \\ g & h \end{pmatrix} - \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}\right] = \det\begin{pmatrix} h-E & g \\ g & h-E \end{pmatrix}$$

$$= (h-E)^2 - g^2 = 0$$

$$(h - E - g)(h - E + g) = 0 \quad \Rightarrow$$

$$E_1 = h + g$$

$$E_2 = h - g$$

To find the corresponding eigenfunctions, we plug our eigenvalues back into

$$H|S\rangle_n = E_n|S\rangle, \quad n = 1, 2 \qquad |S\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

#1

$$\begin{pmatrix} h & g \\ g & h \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (h+g)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} h \cdot a + g \cdot b \\ g \cdot a + h \cdot b \end{pmatrix} = \begin{pmatrix} (h+g)a \\ (h+g)b \end{pmatrix}$$

$$\cancel{ha} + gb = \cancel{ha} + ga \quad \Rightarrow \quad \boxed{a = b}$$

$$ga + \cancel{hb} = \cancel{hb} + gb$$

$$a = b \Rightarrow |s\rangle_1 = \begin{pmatrix} a \\ a \end{pmatrix}$$

Normalization gives $\langle s|s\rangle = 1$ $\quad (a^* \ a^*)\begin{pmatrix} a \\ a \end{pmatrix} = 1$

$$2|a|^2 = 1$$

$$a = \frac{1}{\sqrt{2}} \Rightarrow \boxed{|s\rangle_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$ (let's call it $|s\rangle_+$)

#2 $\quad \begin{pmatrix} h & g \\ g & h \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (h-g)\begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} ha + gb \\ ga + hb \end{pmatrix} = \begin{pmatrix} (h-g)a \\ (h-g)b \end{pmatrix}$$

$$\cancel{ha} + gb = \cancel{ha} - ga \qquad \boxed{a = -b}$$
$$ga + \cancel{hb} = \cancel{hb} - gb$$

$$|s\rangle_2 \equiv |s\rangle_- = \begin{pmatrix} a \\ -a \end{pmatrix}$$

Normalization gives $\langle s|s\rangle = 1 \Rightarrow a = \frac{1}{\sqrt{2}}$

$$\boxed{|s_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Summary: we got eigenfunctions $|s\rangle_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|s_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

corresponding to eigenvalues $E = h \pm g$.

Now we return to our time-dependent Schrödinger equation and its general solution:

$$|S\rangle = \sum_{n=1,2} c_n |S\rangle_n e^{-iE_n t/\hbar}$$

We know $|S\rangle_n$ and E_n but need to find c_n .

As before, we do so using the initial wave function at t = 0: $|S(t=0)\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In this case, we can simply write the representation of $|S(t=0)\rangle$ via $|S\rangle_{\pm}$:

$$|S(t=0)\rangle \equiv |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left\{ \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{|S\rangle_+} + \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{|S\rangle_-} \right\} = \underbrace{\frac{1}{\sqrt{2}}}_{c_1 = \frac{1}{\sqrt{2}}} |S\rangle_+ + \underbrace{\frac{1}{\sqrt{2}}}_{c_2 = \frac{1}{\sqrt{2}}} |S\rangle_-$$

Alternatively, we could find c_n as usual using

$$c_n = \langle S_n | S(t=0)\rangle$$

$$c_1 = \frac{1}{\sqrt{2}}(1 \quad 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \text{, etc.}$$

All we do now is plug in $|S\rangle_n$, c_n , and E_n into our general solution:

$$|S\rangle = \frac{1}{\sqrt{2}} \left(e^{-iE_1 t/h} |S\rangle_+ + e^{-iE_2 t/\hbar} |S_-\rangle \right)$$

$$\boxed{|S\rangle = \frac{1}{\sqrt{2}}\left(e^{-i(h+g)t/\hbar} |S\rangle_+ + e^{-i(h-g)t/\hbar} |S\rangle_- \right)}$$

We can simplify it a bit:

$$|S\rangle = \frac{1}{2} e^{-iht/\hbar} \left[e^{-igt/\hbar} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{igt/\hbar} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

$$= \frac{1}{2} e^{-iht/\hbar} \begin{pmatrix} e^{-igt/\hbar} + e^{igt/\hbar} \\ e^{-igt/\hbar} - e^{igt/\hbar} \end{pmatrix}$$

$$\boxed{|S\rangle = e^{-iht/\hbar} \begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix}} \longleftarrow \text{Our final result.}$$

Note: this problem represents crude model of neutrino oscillation.
State $|1\rangle$ would be electron neutrino and state $|2\rangle$ would represent muon neutrino. If off-diagonal term in the Hamiltonian $g \neq 0$, then electron neutrino can turn with time into muon neutrino and back.

Projection operator

The operator $\hat{P} \equiv |\alpha\rangle\langle\alpha|$, where $|\alpha\rangle$ is a normalized state vector is called projection operator onto the subspace spanned by $|\alpha\rangle$ since it picks out the portion of any other vector that "lies along" $|\alpha\rangle$:

$$\hat{P}|\beta\rangle = \langle\alpha|\beta\rangle|\alpha\rangle.$$

If $\{|e_n\rangle\}$ is a discrete orthonormal basis,

$$\langle e_m|e_n\rangle = \delta_{mn}$$

Then
$$\sum_n |e_n\rangle\langle e_n| = 1$$

and
$$|\alpha\rangle = \sum_n |e_n\rangle\langle e_n|\alpha\rangle.$$

Example 2

Consider three-dimensional vector space spanned by orthonormal basis set $|1\rangle, |2\rangle, |3\rangle$. $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle$$

$$|\beta\rangle = i|1\rangle + 2|3\rangle$$

(a) Construct $\langle\beta|$ (in terms of $\langle 1|, \langle 2|, \langle 3|$).

(b) Find $\langle\beta|\alpha\rangle$.

(c) Find matrix elements A_{11} and A_{12} of the operator $\hat{A} \equiv |\alpha\rangle\langle\beta|$ in this basis.

Solution

(a) $|\beta\rangle = i|1\rangle + 2|3\rangle \Rightarrow \boxed{\langle\beta| = -i\langle 1| + 2\langle 3|}$

(b) $\langle\beta|\alpha\rangle = (-i\langle 1| + 2\langle 3|)(i|1\rangle - 2|2\rangle - i|3\rangle)$

$$= -i \cdot i \underbrace{\langle 1|1\rangle}_{1} - 2i \underbrace{\langle 3|3\rangle}_{1} = 1 - 2i$$

Note: all other terms like $\langle 1|2\rangle = 0$ since $\langle i|j\rangle = \delta_{ij}$ for orthonormal basis set.

(c)

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Example 2

$$\boxed{A_{ij} = \langle i | A | j \rangle} \longleftarrow \text{matrix element } A_{ij}$$

$$A_{11} = \langle 1 | A | 1 \rangle = \langle 1 | \alpha \rangle \langle \beta | 1 \rangle$$

$$A_{12} = \langle 1 | A | 2 \rangle = \langle 1 | \alpha \rangle \langle \beta | 2 \rangle$$

$$A_{11} = \langle 1 | \left(i|1\rangle - 2|2\rangle - i|3\rangle \right) \left(-i\langle 1| + 2\langle 3| \right) |1\rangle$$

$$= \underbrace{i\langle 1|1\rangle}\,(-i)\langle 1|1\rangle = 1$$

All other terms are zero.

$$A_{12} = \langle 1 | \alpha \rangle \langle \beta | 2 \rangle = 0 \quad \text{since} \quad \langle \beta | 2 \rangle = 0$$

Example 3

On the subject of commutators

Show that $[x^n, p] = i\hbar n x^{n-1}$.

Solution

We introduce a trial function f

$$[x^n, p]f = \{x^n p - p x^n\} f$$

$$= x^n \left(-i\hbar \frac{df}{dx}\right) + i\hbar \frac{d}{dx}(x^n f)$$

$$= \cancel{-i\hbar x^n \frac{df}{dx}} + i\hbar n x^{n-1} f + \cancel{i\hbar x^n \frac{df}{dx}}$$

$$= i\hbar n x^{n-1} f \quad \text{for any } f(x).$$

Therefore,

$$\boxed{[x^n, p] = i\hbar n x^{n-1}}$$