**Lecture #15**

## Eigenfunctions of hermitian operators

If the **spectrum is discrete**, then the eigenfunctions are in Hilbert space and correspond to physically realizable states.

It the **spectrum is continuous**, the eigenfunctions are not normalizable, and they do not correspond to physically realizable states (but their linear combinations may be normalizable).

## Discrete spectra

Normalizable eigenfunctions of a hermitian operator have the following properties:

(1) Their eigenvalues are real.
(2) Eigenfunctions that belong to different eigenvalues are orthogonal.

**Finite-dimension vector space:**

The eigenfunctions of an observable operator are **complete**, i.e. any function in Hilbert space can be expressed as their linear combination.

## Continuous spectra

Example: Find the eigenvalues and eigenfunctions of the momentum operator $-i\hbar \frac{d}{dx}$.

$$-i\hbar \frac{d}{dx} f_p(x) = p\, f_p(x)$$

eigenvalue

eigenfunction

Solution:

$$\boxed{f_p(x) = A\, e^{ipx/\hbar}}$$

This solution is not square-inferable and operator p has no eigenfunctions in Hilbert space. However, if we only consider real eigenvalues, we can define sort of orthonormality:

$$\int_{-\infty}^{\infty} f_{p'}^*(x) f_p(x)\, dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\, dx = |A|^2\, 2\pi\hbar\, \delta(p-p')$$

Dirac delta function

since the Fourier transform of Dirac delta function is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

**Proof:**

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx$$

Plancherel's theorem

Now, $f(x) = \delta(x) \Rightarrow F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty} \delta(x)\,e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}$

$$f(x) = \delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

QED

We pick $A = \frac{1}{\sqrt{2\pi\hbar}}$ and get

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\,e^{ipx/\hbar}$$

Then, $\langle f_{p'} | f_p \rangle = \delta(p-p')$ which looks very similar to

orthonormality. We can call such equation Dirac orthonormality.

These functions are complete in a sense that any square integrable function $f(x)$ can be written in a form

$$f(x) = \int_{-\infty}^{\infty} c(p)\,f_p(x)\,dp = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} c(p)\,e^{ipx/\hbar}\,dp$$

and c(p) is obtained by Fourier's trick.

**Summary for continuous spectra:** eigenfunctions with real eigenvalues are Dirac orthonormalizable and complete.

**Generalized statistical interpretation:**

If your measure observable Q on a particle in a state $\Psi(x,t)$ you will get one of the eigenvalues of the hermitian operator $\hat{Q}$. If the spectrum of $\hat{Q}$ is discrete, the probability of getting the eigenvalue $q_n$ associated with orthonormalized eigenfunction $f_n(x)$ is

$$|c_n|^2, \text{ where } \quad c_n = \langle f_n | \Psi \rangle.$$

It the spectrum is continuous, with real eigenvalues q(z) and associated Dirac-orthonormalized eigenfunctions $f_z(x)$, the probability of getting a result in the range dz is

$$|c(z)|^2 dz, \text{ where } \quad c(z) = \langle f_z | \Psi \rangle$$

The wave function "collapses" to the corresponding eigenstate upon measurement.

$$\boxed{\sum_n |c_n|^2 = 1 \quad \text{and} \quad \langle Q \rangle = \sum_n q_n |c_n|^2}$$ Discrete Spectrum

**The uncertainty principle:**

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

$$[A, B] = AB - BA$$

$$\left.\begin{array}{l} \hat{A} = x \\ \hat{B} = p \end{array}\right\} \Rightarrow \quad \sigma_x^2 \sigma_p^2 \geq \left( \frac{1}{2i} i\hbar \right)^2 = \left( \frac{\hbar}{2} \right)^2$$

$$[\hat{x}, \hat{p}] = i\hbar \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Therefore, there is an uncertainty principle for any pair of observables whose operators do not commute.

**Example:**

Imagine a system in which there are just two linearly independent states

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The most general state is a normalized linear combination

$$|S\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

The Hamiltonian can be expressed as a hermitian matrix, suppose it has a form

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix},$$

where h and g are real constants. If the system starts out (at t=0) in state $|1\rangle$ what is its state in time t?