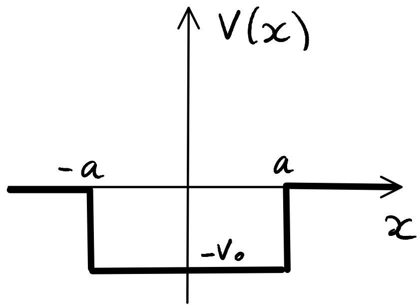
Lecture 13

The finite square well



$$V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & |x| > a \end{cases}$$

Bound states: $E < 0$
Scattering states: $E > 0$

Bound states
Step 1: Solve Schrödinger equation for all regions

① Region $x < -a$ $\quad V = 0$ and $\quad -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} = E\psi \qquad (E<0)$

$$\frac{d^2\psi}{dx^2} = k^2\psi \quad \text{where} \quad k \equiv \frac{\sqrt{-2mE}}{\hbar} \text{ is real \& positive}$$

$$\psi = \cancel{A e^{-kx}} + B e^{kx} \Rightarrow \boxed{\psi(x) = B e^{kx} \quad x < -a}$$

This term blows up $x \to -\infty$

② Region $-a < x < a$ $\quad V(x) = -V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \text{ , or } \quad \frac{d^2\psi}{dx^2} = -\ell^2\psi$$

$$\ell \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar} \quad \text{is real and positive } (E > V_{min})$$

General solution is $\psi = C' e^{i\ell x} + D' e^{-i\ell x}$ but we will use sin/cos form to distinguish even and odd solutions.

$$\boxed{\psi(x) = C\sin(\ell x) + D\cos(\ell x) \quad \text{for} \quad -a < x < a}$$

③ Region $x > a$ $\quad \psi(x) = Fe^{-\kappa x} + \cancel{Ge^{\kappa x}}$ blows up $x \to \infty$

$$\boxed{\psi(x) = Fe^{-\kappa x} \quad x > a}$$

Step 2: Apply boundary conditions that ψ and $d\psi/dx$ are continuous at a and $-a$.

Note: V (x) is even $\Rightarrow$ wave functions are either even or odd. Therefore, we only need to impose conditions on one side, and use $\psi(-x) = \pm\,\psi(x)$ for the other side.

Even solutions:

$$\boxed{\psi(x) = \begin{cases} Fe^{-\kappa x} & x > a \\ D\cos(lx) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}}$$

(1) ψ is continuous at a $\Rightarrow$

$$Fe^{-\kappa a} = D\cos(la) \qquad (1)$$

(2) $\frac{d\psi}{dx}$ is continuous at a

$$-F\kappa e^{-\kappa a} = -lD\sin(la) \qquad (2)$$

We divide equations (2) and (1) to get

$$\frac{+\cancel{F}\kappa \cancel{e^{-\kappa a}}}{\cancel{F}\cancel{e^{-\kappa a}}} = \frac{+l\cancel{D}\sin(la)}{\cancel{D}\cos(la)}$$

$$\boxed{\kappa = l\tan(la)}$$

This equation gives you formula for allowed energies (remember that $k \equiv \frac{\sqrt{-2mE}}{\hbar}$ and $\ell \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$).

One can solve this equation numerically. First, we change variables:

$$z \equiv \ell a; \qquad z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$$

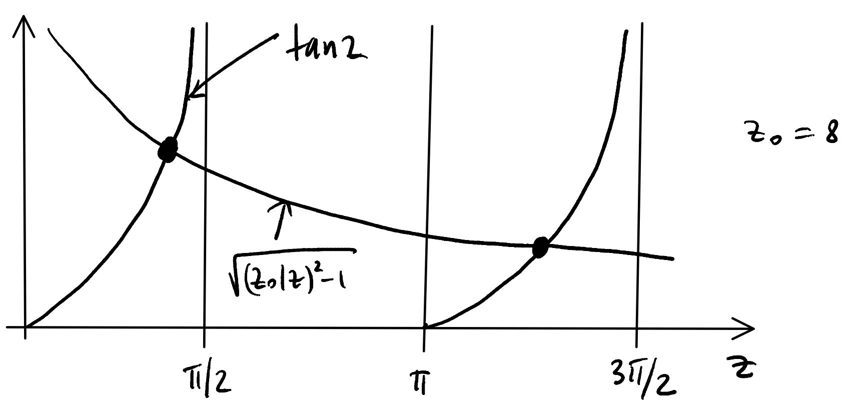
$$\kappa = \ell \tan(\ell a) \Rightarrow \boxed{\tan z = \sqrt{(z_0/z)^2 - 1}}$$

Check: $\frac{\kappa}{\ell} = \sqrt{\frac{z_0^2}{z^2} - 1} = \frac{1}{\ell a}\sqrt{z_0^2 - z^2}$

$$\Rightarrow \kappa a = \sqrt{z_0^2 - z^2} = \sqrt{\frac{a^2}{\hbar^2}2mV_0 - 2m(V_0+E)\frac{a^2}{\hbar^2}} =$$

$$= \frac{a}{\hbar}\sqrt{-2mE} = \kappa a \ (\text{ok})$$

Solutions of equation $\tan z = \sqrt{(z_0/z)^2 - 1}$



Step 3. Normalize ψ (Find D and F) . (Homework).

Scattering states

E>0
Again, step 1: solve Schrödinger equation for all regions.

$x < -a$: $V = 0$: $\psi(x) = Ae^{ikx} + Be^{-ikx}$ $\quad k = \frac{\sqrt{2mE}}{\hbar}$

$-a < x < a$: $V = -V_0$: $\psi(x) = C\sin(lx) + B\cos(lx)$

$$l = \frac{\sqrt{2m(V_0+E)}}{\hbar}$$

$x > a$ (assuming no incoming wave from the right)

$\psi(x) = Fe^{ikx}$ $\quad (G=0)$

A is amplitude of **incoming wave** (from the left).
B is the **reflected amplitude** and **F** is the **transmitted amplitude**.
Since the scattering problem is asymmetric, we use exp notations instead of sin/cos ones.

Step 2. Apply boundary conditions.

There are four boundary conditions:

(1) ψ is continuous at a.
(2) ψ is continuous at $-a$.
(3) $d\psi/dx$ is continuous at a.
(4) $d\psi/dx$ is continuous at $-a$

$$\boxed{\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ C\sin(lx) + D\cos(lx) & -a < x < a \\ Fe^{ikx} & x > a \end{cases}}$$

$\psi(a)$ (1) gives: $C\sin(\ell a) + D\cos(\ell a) = Fe^{ika}$

$\psi(-a)$ (2) gives: $Ae^{-ika} + Be^{ika} = -C\sin(\ell a) + D\cos(\ell a)$

$\left.\frac{d\psi}{dx}\right|_a$ (3) gives: $\ell[C\cos(\ell a) - D\sin(\ell a)] = ikFe^{ika}$

$\left.\frac{d\psi}{dx}\right|_{-a}$ (4) gives: $ik[Ae^{-ika} - Be^{ika}] = \ell[C\cos(\ell a) + D\sin(\ell a)]$

We can use (1) and (3) to eliminate C & D:
and solve remaining for B and F:

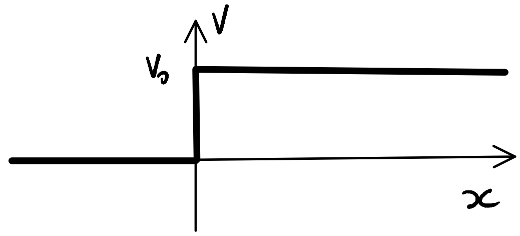
$$B = i\frac{\sin(2\ell a)}{2k\ell}(\ell^2 - k^2)F$$

$$F = \frac{e^{-2ika}\,A}{\cos(2\ell a) - i\frac{(k^2+\ell^2)}{2k\ell}\sin(2\ell a)}$$

Transmission coefficient $T = \frac{|F|^2}{|A^2|}$

Exercise 6 (homework problem)

Consider "step" potential



$$V(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ V_0 & \text{if} \quad x > 0 \end{cases}$$

(a) Calculate the reflection coefficient, for E < V_0 and comment on the result.

(b) Calculate the reflection coefficient for the case E > V_0.

① step 1 $\quad x < 0 \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\boxed{\psi(x) = A e^{ikx} + B e^{-ikx}}$$

$x > 0 \qquad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$

$$\frac{d^2\psi}{dx^2} = (V_0 - E)\frac{2m}{\hbar}\psi = \ell^2\psi$$

$$\ell = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \qquad V_0 > E$$

$$\psi = F e^{-\ell x} + \cancel{G e^{\ell x}}$$

blows up for $x = \infty$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x<0) \\ Fe^{-\ell x} & (x>0) \end{cases}$$

Step2. Boundary conditions

ψ is continuous at $x=0 \Rightarrow A+B=F$

$d\psi/dx$ is cont. at $x=0 \Rightarrow ik(A-B) = -\ell F$

$$ik(A-B) = -\ell(A+B)$$

$$A\left(1+\frac{ik}{\ell}\right) = -B\left(1-\frac{ik}{\ell}\right)$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1+ik/\ell)|^2}{|(1-ik/\ell)|^2} = \frac{(1-ik/\ell)(1+ik/\ell)}{(1+ik/\ell)(1-ik/\ell)} = 1$$

Wave function penetrates into the barrier, but eventually it is all reflected.

(b) $\psi = \begin{cases} Ae^{ikx} + Be^{ikx} \\ Fe^{i\ell x} \end{cases}$ $\quad k = \frac{\sqrt{2mE}}{\hbar}, \quad \ell = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

(1) $A+B=F$

(2) $ik(A-B) = i\ell F$

$$A+B = \frac{k}{\ell}(A-B), \quad A\left(1-\frac{k}{\ell}\right) = -B\left(1+\frac{k}{\ell}\right),$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(k-\ell)^4}{(k^2-\ell^2)^2} = \frac{(\sqrt{E}-\sqrt{E-V_0})^4}{V_0^2}$$