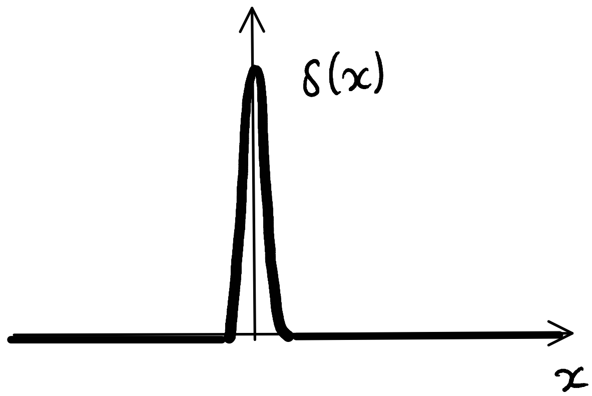
Lecture 12

Review: Dirac delta-function



It is infinitely high, infinitesimally narrow spike whose area is 1.

$$\delta(x) \equiv \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Property of the delta-function:

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

This is the most important property of the delta-function:

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)}$$

<u>Class exercise 5</u>

Evaluate the following integrals:

(a) $\displaystyle\int_{-3}^{+1} (x^3 - 3x^2 + 2x - 1)\,\delta(x+2)\,dx$

(b) $\displaystyle\int_{-1}^{+1} \exp(|x| + 3)\,\delta(x-2)\,dx$

Class exercise 5

Evaluate the following integrals:

(a) $$\int_{-3}^{+1} (x^3 - 3x^2 + 2x - 1)\, \delta(x+2)\, dx$$

(b) $$\int_{-1}^{+1} \exp(|x| + 3)\, \delta(x-2)\, dx$$

Solution

(a) $$\int_{-3}^{+1} \underbrace{(x^3 - 3x^2 + 2x - 1)}_{f(x)}\, \underbrace{\delta(x+2)}_{\delta(x+a)}\, dx =$$

$$= (-2)^3 - 3(-2)^2 + 2(-2) - 1 = -25$$

where we used

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a).$$

(b) $$\int_{-1}^{+1} \exp(|x| + 3)\, \delta(x-2)\, dx = 0$$

since x = 2 is outside of the domain of integration.

Delta-function potential:

$$V(x) = -\alpha\delta(x)$$

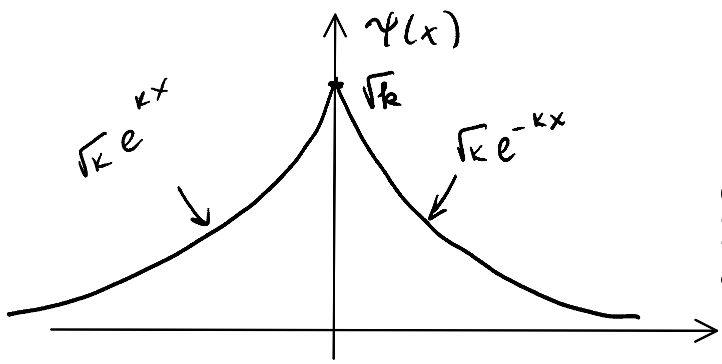
some positive constant

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi = E\psi$$

Bound states $(E<0)$
Scattering states $(E>0)$

<u>We found the bound states:</u>

$$\psi(x) = \begin{cases} \sqrt{\kappa}\, e^{\kappa x} & (x \le 0) \\ \sqrt{\kappa}\, e^{-\kappa x} & (x \geq 0) \end{cases}$$



However, we did not find the energies or how many bound states we have.
In fact, we did not use the delta-function at all!

We are going to integrate the Schrödinger equation from $-\varepsilon$ to $+\varepsilon$ and take the limit at $\varepsilon \to 0$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\underbrace{\int_{-\varepsilon}^{+\varepsilon}\frac{d^2\psi}{dx^2}\,dx}_{\left.\frac{d\psi}{dx}\right|_{-\varepsilon}^{\varepsilon}} - \int_{-\varepsilon}^{\varepsilon}\alpha\,\delta(x)\,\psi(x)\,dx = \underbrace{\int_{-\varepsilon}^{\varepsilon}E\psi(x)\,dx}$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}E\psi\,dx = 0$$ (area with width $\to 0$ and finite height)

$$\Delta\left(\frac{d\psi}{dx}\right) = \lim_{\varepsilon\to 0} \frac{d\psi}{dx}\Bigg|_{-\varepsilon}^{+\varepsilon} = \frac{d\psi}{dx}\Bigg|_{+} - \frac{d\psi}{dx}\Bigg|_{-} = -\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\varepsilon} \alpha\,\delta(x)\,\psi(x)\,dx$$

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\,\psi(0)$$

$\sqrt{\kappa}$

We now determine $\Delta\left(\frac{d\psi}{dx}\right)$.

$$\psi(x) = \begin{cases} \sqrt{\kappa}\, e^{\kappa x} \quad (x \le 0) & d\psi/dx = \kappa\sqrt{\kappa}\, e^{\kappa x} \qquad d\psi/dx\big|_{-} = \kappa\sqrt{\kappa} \\ \sqrt{\kappa}\, e^{-\kappa x} \quad (x \ge 0) & d\psi/dx = -\kappa\sqrt{\kappa}\, e^{\kappa x} \qquad d\psi/dx\big|_{+} = -\kappa\sqrt{\kappa} \end{cases}$$

$$-\kappa\sqrt{\kappa} - \kappa\sqrt{\kappa} = -\frac{2m\alpha}{\hbar^2}\sqrt{\kappa} \quad \Rightarrow$$

$$\boxed{\kappa = \frac{m\alpha}{\hbar^2}} \qquad\qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

Allowed energy is $\quad E = -\dfrac{\hbar^2\kappa^2}{2m} = -\dfrac{m\alpha^2}{2\hbar^2} \Rightarrow$

The delta-function potential has only one bound state:

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\, e^{-m\alpha|x|/\hbar^2} \;;\quad E = -\frac{m\alpha^2}{2\hbar^2}$$

Note: $\quad x > 0 \Rightarrow e^{-\kappa x}$ or $e^{-\kappa|x|}$

$\qquad\quad x < 0 \Rightarrow e^{\kappa x} \rightarrow e^{-\kappa|x|}$ (OK)

Scattering states (E > 0)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \qquad k \equiv \frac{\sqrt{2mE}}{\hbar} \text{ is real and positive}$$

$$x<0 \qquad \psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$x>0 \qquad \psi(x) = Fe^{ikx} + Ge^{-ikx}$$

$\psi(x)$ has to be continuous at $x=0 \Rightarrow$

$$\boxed{A + B = F + G}$$

Now we use $\quad \Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\underbrace{\psi(0)}_{\psi(0) = A+B}$

The derivatives are

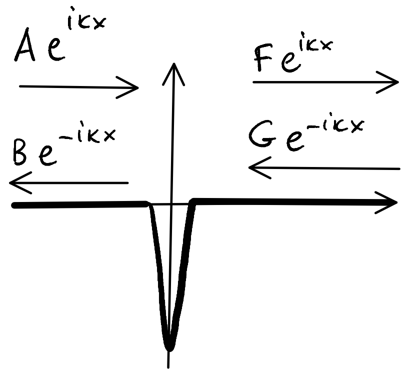
$$\begin{cases} x>0 & d\psi/dx = ik(Fe^{ikx} - Ge^{-ikx}), \quad d\psi/dx|_+ = ik(F-G) \\ x<0 & d\psi/dx = ik(Ae^{ikx} - Be^{-ikx}), \quad d\psi/dx|_- = ik(A-B) \end{cases}$$

$$\boxed{ik(F-G) - ik(A-B) = -\frac{2m\alpha}{\hbar^2}(A+B)}$$

Note: these are not normalizable states.

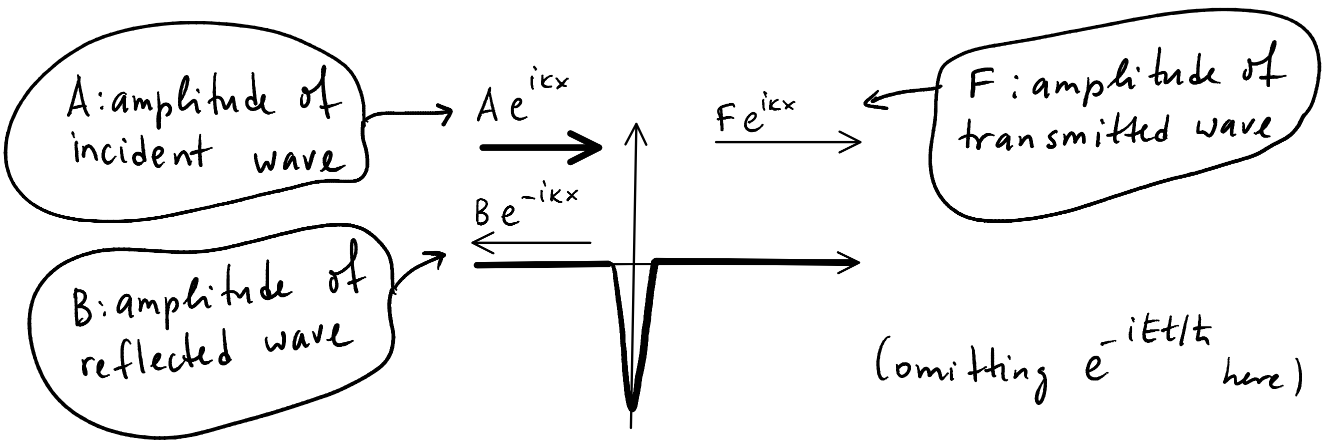
Interpretation of our solution:

$x<0 \quad \psi = Ae^{ikx} + Be^{-ikx}$ $\qquad\qquad$ $x>0 \quad \psi = Fe^{ikx} + Ge^{-ikx}$



(omitting $e^{-iEt/\hbar}$ here)

Typical scattering experiment: particles are fired from one direction. If they are coming from the left, then G=0 (amplitude of particles coming from the right).



Returning to our equations

$$F + \underbrace{\cancel{G}}_{=0} = A + B$$

$$ik(F - \underbrace{\cancel{G}}_{=0} - A + B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

we can find B and F in terms of A.

Introducing designation $\beta \equiv \frac{m\alpha}{\hbar^2 k}$ we get

$$\begin{cases} F = A + B \\ F - A + B = \frac{2im\alpha}{k\hbar^2}(A+B) = 2i\beta(A+B) \end{cases}$$

$$\cancel{A} + B - \cancel{A} + B = 2i\beta(A+B)$$

$$B = i\beta A + i\beta B$$

$$B(1 - i\beta) = i\beta A$$

$$\boxed{B = \frac{i\beta}{1 - i\beta} A}$$

amplitude of the reflected wave

$$F = A + B = A + \frac{i\beta}{1-i\beta} A = \frac{1 - \cancel{i\beta} + \cancel{i\beta}}{1 - i\beta} A$$

$$\boxed{F = \frac{1}{1 - i\beta} A}$$

amplitude of the transmitted wave

Reflection coefficient R: relative probability that incident particle will be reflected back.

$$\boxed{R = \frac{|B|^2}{|A|^2}} = \frac{\beta^2}{1 + \beta^2}$$

$$\beta = \frac{m\alpha}{\hbar^2 k} \quad \text{and} \quad k = \frac{\sqrt{2mE}}{\hbar} \Rightarrow \beta^2 = \frac{m^{\cancel{2}}\alpha^2\cancel{\hbar^2}}{\hbar^{\cancel{4}2}\, 2\cancel{m}E} = \frac{m\alpha^2}{2E\hbar^2}$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} = \frac{1}{1+\frac{2\hbar^2 E}{m\alpha^2}}$$

The relative probability of transmission is given by the transmission coefficient T defined as

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

Sum of R and T has to be 1.

$$\boxed{R + T = 1}$$

$$T = \frac{1}{1+\beta^2} = \frac{1}{1+(m\alpha^2/2\hbar^2 E)}$$

The higher the energy, the larger is transmission coefficient.

As we already know, the solution to the problem that our stationary states are not normalizable is to form linear combinations of states as we did for the free particle. The true particles are represented by the wave packets (involving range of energies).

Rand T are then interpreted as approximate reflection and transmission coefficients for particles with energies in the vicinity of E.