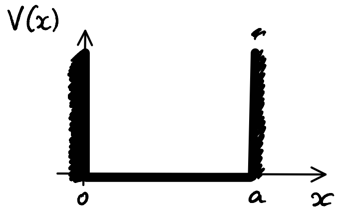
Lecture 12. Problem solving

Problem 1

A particle of mass m is in the ground state (n=1) of the infinite square well:



$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$

Solutions: $$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right), \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 1, 2, \ldots$$

Suddenly the well expands to twice its original size - the right wall moving from a to 2a leaving the wave function (momentarily) undisturbed. The energy of particle in now measured. What is the probability of getting the result

$$E = \frac{\pi^2 \hbar^2}{2ma^2}$$

(same as the initial energy)?

Solution:

$$\psi(x, 0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} x\right)$$

After the wall expands, the new states and energies are:

$$\psi_n(x) = \sqrt{\frac{2}{(2a)}} \sin\left(\frac{n\pi}{2a} x\right) \; ; \; E_n = \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2} \Rightarrow$$

The result $E = \dfrac{\pi^2 \hbar^2}{2ma^2}$ corresponds to $n = 2$:

$$E_2 = \frac{4\pi^2 \hbar^2}{2m\, 4a^2} = \frac{\pi^2 \hbar^2}{2ma^2} .$$

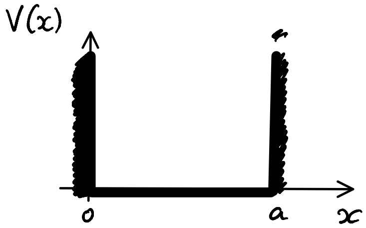
Therefore, the probability of getting this result is given by $P = |c_2|^2$.

$$c_2 = \int \psi_2^*(x)\, \psi(x, 0)\, dx = \int_0^a \sqrt{\frac{2}{2a}} \sin\left(\frac{2\pi}{2a} x\right) \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} x\right) dx$$

$$= \frac{\sqrt{2}}{a} \int_0^a \sin^2\left(\frac{\pi}{a} x\right) dx = \frac{1}{\sqrt{2}} \Rightarrow \boxed{P = \frac{1}{2}}$$

Problem 2

Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$ for the nth stationary state of the infinite square well.



$$V(x) = \begin{cases} 0, & \text{if} \quad 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$

Solutions: $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$, $E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$, $n = 1, 2, \ldots$

Solution

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x,t)|^2 dx = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx$$

$$= \int_0^a \frac{2}{a} x \sin^2\left(\frac{\pi n}{a} x\right) dx = \frac{a}{2}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx = a^2 \left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right].$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = m \frac{d}{dt}\left(\frac{a}{2}\right) = 0$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi_n^* \left((-i\hbar)^2 \frac{d^2}{dx^2} \right) \psi_n \, dx$$

$$\boxed{\text{in our case} \quad -\frac{\hbar^2}{2m} \frac{d^2\psi_n}{dx^2} = E_n \psi_n} \Rightarrow$$

$$= \int_{-\infty}^{\infty} \psi_n^* (2mE_n) \psi_n \, dx = 2mE_n \underbrace{\int_{-\infty}^{\infty} \psi_n^* \psi_n \, dx}_{=1} = 2mE_n$$

$$= \cancel{2m} \left(\frac{\pi^2 \hbar^2 n^2}{\cancel{2m} a^2} \right) = \left(\frac{\pi \hbar n}{a} \right)^2$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{a^2 \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{2(n\pi)^2} \right)}$$

$$= \frac{a}{2} \sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\pi n \hbar}{a}$$

$\sigma_x \sigma_p = \frac{\hbar}{2} \sqrt{\frac{(n\pi)^2}{3} - 2}$, *it is the smallest for*

$n = 1$, $\sigma_x \sigma_p = \frac{\hbar}{2} (1.14) > \frac{\hbar}{2}$

Problem 3

A particle of mass m in the harmonic oscillator potential starts out in the state

$$\psi(x,0) = A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}\,x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

for some constant A.

What is the expectation value of the energy?

Harmonic oscillator

First three states are $\psi_0(x) = \alpha e^{-\xi^2/2}$

$$\psi_1(x) = \sqrt{2}\,\alpha\,\xi\, e^{-\xi^2/2}$$

$$\psi_2(x) = \frac{\alpha}{\sqrt{2}}\left(2\xi^2 - 1\right) e^{-\xi^2/2}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\,x \qquad \alpha \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Energies: $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

Solution

$$\psi(x,0) = A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}\,x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A\left(1 - 2\xi\right)^2 e^{-\xi^2/2}$$

$$= A\left(1 - 4\xi + 4\xi^2\right) e^{-\xi^2/2}$$

This function can be expressed as a linear combination of the first three states of harmonic oscillator.

$$\boxed{\psi(x,0) = c_0\,\psi_0(x) + c_1\,\psi_1(x) + c_2\,\psi_2(x)}$$

Now, we need to find coefficients c by equating same powers of ξ :

$$\Psi(x,0) = (A - 4A\,\xi + 4A\,\xi^2)\, e^{-\xi^2/2}$$

$$\psi(x,0) = \left(\alpha c_0 + \sqrt{2}\,\alpha c_1\, \xi + \frac{\alpha}{\sqrt{2}} \cdot 2 c_2 \xi^2 - \frac{\alpha}{\sqrt{2}} c_2\right) e^{-\xi^2/2}$$

$$\xi^0: \quad A = \alpha c_0 - \frac{\alpha}{\sqrt{2}} c_2 = \alpha c_0 - \frac{\cancel{\alpha}}{\cancel{\sqrt{2}}}\, 2 \cdot \cancel{\sqrt{2}}\, \frac{A}{\cancel{\alpha}} = \alpha c_0 - 2A$$

$$c_0 = 3A/\alpha$$

$$\xi: \quad -4A = \sqrt{2}\,\alpha c_1 \quad \Rightarrow \quad c_1 = -2\sqrt{2}\,A/\alpha$$

$$\xi^2: \quad 4A = \alpha\sqrt{2}\, c_2 \quad \Rightarrow \quad c_2 = \frac{2\sqrt{2}\,A}{\alpha}$$

Normalization gives: $1 = |c_0|^2 + |c_1|^2 + |c_2|^2$

$$= 9\frac{A^2}{\alpha^2} + 8\frac{A^2}{\alpha^2} + 8\frac{A^2}{\alpha^2} = 25\frac{A^2}{\alpha^2} \Rightarrow \boxed{A = \frac{\alpha}{5}}$$

$$c_0 = \frac{3}{5}, \quad c_1 = -\frac{2\sqrt{2}}{5}, \quad c_2 = \frac{2\sqrt{2}}{5}$$

Now it is really easy to find the expectation value of energy:

$$\langle H \rangle = \sum_n |c_n|^2 E_n = \sum_n |c_n|^2 \left(n + \frac{1}{2}\right)\hbar\omega$$

Proof: $H\psi_n = E_n \psi_n$

$$\langle H \rangle = \int \psi^* H \psi\, dx = \int \left(\sum_m c_m \psi_m^*\right) H \left(\sum_n c_n \psi_n\right) dx$$

$$= \sum_m \sum_n c_m^* c_n E_n \int \psi_m^* \psi_n\, dx = \sum_n |c_n|^2 E_n$$

$$\langle H \rangle = C_0^2\left(\frac{1}{2}\hbar\omega\right) + C_1^2\left(\frac{3}{2}\hbar\omega\right) + C_2^2\left(\frac{5}{2}\hbar\omega\right)$$

$$= \frac{9}{25}\left(\frac{1}{2}\hbar\omega\right) + \left(\frac{8}{25}\right)\left(\frac{3}{2}\hbar\omega\right) + \left(\frac{8}{25}\right)\left(\frac{3}{2}\hbar\omega\right)$$

$$\boxed{\langle H \rangle = \frac{73}{50}\hbar\omega}$$

Mathematical formulas

$$\int_0^a \sin^2 \frac{\pi x}{a} \, dx = \frac{a}{2}$$

$$\int_0^a x \sin^2 \left(\frac{\pi n x}{a} \right) dx = \frac{a^2}{4}$$

$$\int_0^a x^2 \sin^2 \left(\frac{\pi n x}{a} \right) dx = \frac{a^3}{2} \left(\frac{1}{3} - \frac{1}{2\pi^2 n^2} \right)$$

for integer n